Section 2.1: Discrete Random Variables

Section 2.2: Mathematical Expectation

Section 2.3: The Mean, Variance, and Standard Deviation

Section 2.4: Bernoulli Trials and the Binomial Distribution

Section 2.5: The Moment-Generating Function

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Random Variables

Definition
If $A$ and $B$ are sets, and $f$ is a function from $A$ to $B$, we write

$$f : A \rightarrow B.$$ 

Definition
A random variable $X$ is a function from the sample space $S$ to the real numbers.

$$X : S \rightarrow \mathbb{R}$$

Example
Two coins are flipped and the resulting sequence of heads/tails is noted. Let $X$ be the number of heads in the sequence.
Example

Assuming the coins are fair and independent, calculate $P(X = 1)$ and $P(X \geq 1)$.

- $(X = 1)$ is shorthand for $\{s \in S \mid X(s) = 1\} = \{HT, TH\}$
- $(X \geq 1)$ is shorthand for $\{s \in S \mid X(s) \geq 1\} = \{HT, TH, HH\}$

Definition

If $A \subseteq \mathbb{R}$, define the event

$$(X \in A) = \{s \in S \mid X(s) \in A\}.$$
Example

Roll two independent fair dice, and let \( X \) be the sum of the rolls. Calculate \( P(X = x) \), for \( x = 5, 6, 7, 8, 9 \).

Definition

The *support* of a random variable \( X \) is the set of possible values of \( X \),

\[
\text{supp}(X) = \{X(s) \mid s \in S\}
\]

Definition

A random variable is called *discrete* if its support is countable (is finite or can be put in one-to-one correspondence with the positive integers).
Example

A fair coin is flipped until the result is heads, and $X$ is the number of flips that occur.

- What is the support of $X$?
- Is $X$ a discrete random variable?

Definition

The *probability mass function* $f$ of a discrete random variable $X$ is

$$f : \mathbb{R} \rightarrow [0, 1]$$

$$f(x) = P(X = x)$$

- abbreviated p.m.f.
- also called the probability *distribution* function or probability *density* function (p.d.f.)
Proposition

A function \( f : \mathbb{R} \rightarrow [0, 1] \) is the p.m.f. of some random variable if and only if

- \( f(x) \geq 0 \), for \( x \in \mathbb{R} \), and
- \[ \sum_{x \in \mathbb{R}} f(x) = 1. \]

Example

Let \( X \) be a random variable with p.m.f.

\[ f(x) = \begin{cases} 
  cx^2, & x = 1, 2, 3, 4, 5 \\
  0 & \text{otherwise.}
\end{cases} \]

Find \( c \).
Example

Let $X$ be the number of aces in a five-card poker hand.
- Find the p.m.f. of $X$.
- Draw a probability histogram for $X$.
- The number of aces in each of ten poker hands is listed below:

\[0, 0, 0, 1, 0, 0, 2, 0, 1, 1\]

Draw a relative frequency histogram for this data on the same set of axes as the probability histogram.
Definition (Hypergeometric Distribution)

Setting:
- Set of objects of two types
- $N = \text{total number of objects}$
- $N_1 = \text{number of objects of the 1st type}$
- $N_2 = \text{number of objects of the 2nd type}$
- Select $n$ objects randomly without replacement
- $X = \text{number of objects in sample of 1st type}$
- The p.m.f. of $X$ is

$$P(X = x) = \binom{N_1}{x} \binom{N_2}{n-x} \binom{N}{n}.$$

$X$ is said to have a \textit{hypergeometric distribution}.
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Example

When you buy a scratch-off lottery ticket, you have an 80% chance of winning nothing, a 15% chance of winning $2, and a 5% chance of winning $10. If the ticket costs $1, should you buy one?

Definition

Suppose $X$ is a discrete random variable with p.m.f. $f$. Then the expected value of $X$ is

$$E(X) = \sum_{x \in \mathbb{R}} xf(x),$$

assuming the series converges absolutely. Otherwise, the expected value does not exist.
Example

Let $X$ be the number of heads occurring when a fair coin is flipped 3 times.

- What is the expected value of $X$?
- Find $E(X^2 + 7X)$
- Find $E(5X + 4)$

Suppose $u : \mathbb{R} \to \mathbb{R}$.

$$E(u(X)) = \sum_{x \in \mathbb{R}} u(x)f(x)$$

For random variables $X$ and $Y$ and a constant $c$,

- $E(X + Y) = E(X) + E(Y)$
- $E(cX) = cE(X)$
- $E(c) = c$
An automobile insurance policy has a deductible of $500. Let $X$ be the cost of damages to a vehicle in an accident, and assume $X$ has the following p.m.f.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>250</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>0.1</td>
<td>0.2</td>
<td>0.4</td>
<td>0.2</td>
<td>0.1</td>
</tr>
</tbody>
</table>

If an accident occurs, what is the expected value of the payment made by the insurance company?
**Example**

In a club with 100 members, 60 members approve of the president. In a random sample of size 5, let $X$ be the number of people who approve of the mayor. Find the expected value of $X$.

Let $X$ have a hypergeometric distribution where

- $N_1 =$ number of objects of type 1
- $N =$ total number of objects
- $n =$ sample size

$$E(X) = \frac{N_1}{N} n$$
Let $X$ have a hypergeometric distribution where

- $N_1 =$ number of objects of type 1
- $N =$ total number of objects
- $n =$ sample size

$$E(X) = \frac{N_1}{N} n$$

Example

If you have 10 red pens and 4 blue pens, and you select 6 pens at random, what is the expected value of the number of blue pens in your sample?
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- $E(X) = \text{expected value of } X$
- $E(X) = \text{“average value” of } X$
- $E(X)$ also called the \textit{mean} of $X$
- Alternative notation:

$$\mu = E(X) \text{ or } \mu_X = E(X)$$

**Definition**

The \textit{variance} of $X$ is

$$\text{Var}(X) = E[(X - \mu)^2] = E(X^2) - \mu^2$$

The \textit{standard deviation} of $X$ is

$$\sigma = \sqrt{\text{Var}(X)}.$$
Definition

The $r$th moment of $X$ about $b$ is

$$E[(X - b)^r].$$

The $r$th moment of $X$ about the origin, $E(X^r)$, is usually just called the $r$th moment of $X$. 

(Tarleton State University)
Definition

Let $x_1, x_2, \ldots, x_n$ be a sample.

- The **sample mean** is
  \[ \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i. \]

- The **sample variance** is
  \[ s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2. \]

- The sample variance can be computed more easily as follows:
  \[ s^2 = \frac{\sum_{i=1}^{n} x_i^2 - \frac{1}{n} (\sum_{i=1}^{n} x_i)^2}{n-1}. \]

- The **sample standard deviation** is $s = \sqrt{s^2}$. 

The variance of a hypergeometric random variable is

\[ \text{Var}(X) = np(1 - p) \frac{N - n}{N - 1}, \]

where \( p = \frac{N_1}{N}. \)
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Definition

- A Bernoulli trial is a random experiment that only has 2 possible outcomes.
- Sample space: \( S = \{ \text{success}, \text{failure} \} \)
- Suppose \( X(\text{success}) = 1 \) and \( X(\text{failure}) = 0 \).
- p.m.f. for \( X \):
  \[
  f(x) = \begin{cases} 
  p, & x = 1 \\
  1 - p, & x = 0 
  \end{cases}
  \]
- \( X \) has a Bernoulli distribution with parameter \( p \).
- \( E(X) = p \)
- \( \text{Var}(X) = p(1 - p) \)
- \( \sigma_X = \sqrt{p(1 - p)} \)
- Alternative notation: \( q = 1 - p \)
Definition

Consider a sequence of Bernoulli trials such that
- \( n \) = the number of trials
- \( p \) = the probability of success on each trial
- the trials are independent
- \( X \) = number of success that occur

\( X \) has a \textit{binomial distribution with parameters} \( n \) and \( p \).

\( X \sim b(n, p) \)

\text{p.m.f. for } X:

\[
f(x) = \binom{n}{x} p^x (1 - p)^{n-x}, \quad x = 0, 1, \ldots, n.
\]

\( E(X) = np \)

\( \text{Var}(X) = np(1 - p) \)
Definition

The *cumulative distribution function* of $X$ is

$$F(x) = P(X \leq x).$$

Often, it is simply called the *distribution function* of $X$.

Example

There is a 15% chance that items produced in a certain factory are defective. Assuming that 9 items are produced, and assuming that they are statistically independent, what is the probability that

- at most 4 are defective?
- at least 6 are defective?
- more than 6 are defective?
- the number of defective items is between 2 and 5 inclusive?
Connection Between the Hypergeometric and Binomial Distributions

Random Sampling
- Without replacement: hypergeometric
- With replacement: binomial

Example
In a university organization with 200 members, 60 are seniors. In a random sample of size 10, what is the probability that 4 are seniors, if the sampling is done
- without replacement?
- with replacement?

Find the expected value, variance, and standard deviation of the number of seniors in the sample under both types of sampling.
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Definition

The *moment-generating function* of $X$ is

$$M(t) = E(e^{tX}),$$

assuming $E(e^{tX})$ is finite on some open interval $-h < t < h$.

Example

Let $X$ be a random variable with p.m.f.

$$f(x) = \frac{1}{14}x^2, \text{ for } x = 1, 2, 3.$$  

Find the moment generating function of $X$.

Example

- If the p.m.f. of $X$ is $f(x) = \frac{6}{\pi^2x^2}$, for $x = 1, 2, \ldots$
- then $X$ does not have a moment generating function.
Example
- Suppose the m.g.f. of $X$ is $M(t) = \frac{3}{6} e^t + \frac{2}{6} e^{2t} + \frac{1}{6} e^{3t}$.
- Find the p.m.f. of $X$.

Example
Find the p.m.f. of $X$ if the m.g.f. is

$$M(t) = \frac{e^{t/2}}{1 - e^{t/2}}, \quad t < \ln(2).$$

Theorem
$X$ and $Y$ have the same m.g.f. if and only if they have the same p.m.f.
- $E(X) = M'(0)$
- $E(X^2) = M''(0)$
- $E(X^r) = M^{(r)}(0)$
- $\text{Var}(X) = M''(0) - M'(0)^2$
Discrete Random Variables
- Definitions of random variables, discrete random variables, p.m.f., and support.
- Probabilities involving random variables
- Properties of a p.m.f.
- Hypergeometric distribution

Mathematical Expectation
- Definition
- Calculating $E(X)$
- Properties of $E$
- Expected value hypergeometric random variable: $E(X) = np$
The Mean, Variance, and Standard Deviation

- Definitions/notation for mean, variance, and standard deviation for random variables and samples
- Shortcut formulas for variance of a random variable/sample
- Be able to compute everything “by hand”.
- Variance of a hypergeometric random variable:
  \[ \text{Var}(X) = np(1 - p) \frac{N-n}{N-1}. \]
- Property of variance: \( \text{Var}(aX + b) = a^2 \text{Var}(X) \), if \( a \) and \( b \) are constants.

Bernoulli Trials and the Binomial Distribution

- Probabilities
- p.m.f.
- c.d.f. table/computer/calculator
- \( E(X) = np, \Var(X) = np(1 - p) \)
Moment-Generating Functions

- p.m.f. $\rightarrow$ m.g.f.
- m.g.f. $\rightarrow$ p.m.f.
- $E(X^r) = M^{(r)}(0)$
- $\text{Var}(X) = M''(0) - M'(0)^2$
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Approximate Poisson Process with Parameter $\lambda > 0$

Setting

- Measuring occurrences of some event on a continuous interval.
- Examples:
  - Number of phone calls received in 1 hour
  - Number of defects in 1 meter of wire

Assumptions

- Occurrences in non-overlapping intervals are independent.
- In a sufficiently short interval of length $h$, the probability of 1 occurrence is approximately $\lambda h$.
- In a sufficiently short interval, the probability of 2 or more occurrences is essentially zero.
Poisson Distribution

- Let \( X = \) \# of occurrences in an interval of length 1
- Then \( X \) has a \textit{Poisson distribution} with parameter \( \lambda \).

\[
f(x) = P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}
\]

Example

Phone calls received by a company are a Poisson process with parameter \( \lambda = 4 \). In a 1 minute period, find the probability of receiving

- 2 calls?
- 5 calls?
- at most 3 calls?
- at least 7 calls?
If $X$ has a Poisson distribution with parameter $\lambda$, then

$E(X) = \text{Var}(X) = \lambda$

For a Poisson process with parameter $\lambda$,

$\lambda$ is the average # of occurrences in an interval of length 1.

**Example**

Phone calls received by a company are a Poisson process, and the company receives an average of 4 calls per minute. In a 3 minute period, find the probability of the company receiving

- 10 calls?
- at most 15 calls?
Interval of Length $t$

- Consider a Poisson process with parameter $\lambda$
- If $X = \#$ of occurrences in an interval of length $t$,
- then $X$ has a Poisson distribution with mean $\lambda t$.

Example

On average, there are 3 flaws in 8 meters of copper wire. For a piece of wire 20 meters long, find the probability of observing

- 5 flaws.
- fewer than 9 flaws.
- Find the expected value, variance, and standard deviation of the number of flaws on a 20 meter piece of wire.
Example

Let \( X \) equal the number of green m&m’s in a package of size 22. Forty-five observations of \( X \) yielded the following frequencies for the possible outcomes of \( X \):

<table>
<thead>
<tr>
<th>Outcome (x):</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency:</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>8</td>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

- Calculate \( \bar{x} \) and \( s^2 \). Are they close?
- Compare the relative frequency histogram to the probability histogram of a Poisson random variable with mean \( \lambda = 5 \).
- Do these data appear to be observations from a Poisson random variable?
If $n$ is large and $p$ is small,

\[
\text{bin}(n, p) \approx \text{Poisson with } \lambda = np.
\]

$n \geq 100$ and $p \leq 0.1$

$n \geq 20$ and $p \leq 0.05$

**Example**

In a shipment of 2000 items, 4% are defective. In a random sample of size 100, find the approximate probability that more than 10 items are defective.