Setting

- $\mathbb{F}$ denotes $\mathbb{R}$ or $\mathbb{C}$
- $V$ is a finite-dimensional, nonzero vector space over $\mathbb{F}$
Outline

1. Generalized Eigenvectors
2. The Characteristic Polynomial
3. Decomposition of an Operator
4. Square Roots
5. The Minimal Polynomial
6. Jordan Form
**Generalized Eigenvectors**

**Definition**

- Suppose $T \in \mathcal{L}(V)$, and $\lambda$ is an eigenvalue of $T$.
- Then $v \in V$ is a **generalized eigenvector** of $T$ corresponding to $\lambda$ if

$$\quad (T - \lambda I)^j v = 0,$$

for some positive integer $j$.

**Example**

- Let $T \in \mathcal{L}(\mathbb{C}^3)$ be defined by $T(z_1, z_2, z_3) = (z_2, 0, z_3)$.
- 0 is an eigenvalue, and the set of generalized eigenvectors corresponding to 0 is

$$\{(z_1, z_2, 0) \mid z_1, z_2 \in \mathbb{C}\}.$$
Example

- Let \( T \in \mathcal{L}(\mathbb{C}^3) \) be defined by \( T(z_1, z_2, z_3) = (z_2, 0, z_3) \).
- 0 is an eigenvalue, and the set of generalized eigenvectors corresponding to 0 is
  \[
  \{(z_1, z_2, 0) \mid z_1, z_2 \in \mathbb{C}\}.
  \]
- 1 is an eigenvalue, and the set of generalized eigenvectors corresponding to 1 is
  \[
  \{(0, 0, z_3) \mid z_3 \in \mathbb{C}\}.
  \]
- \( \mathbb{C}^3 = \{(z_1, z_2, 0) \mid z_1, z_2 \in \mathbb{C}\} \oplus \{(0, 0, z_3) \mid z_3 \in \mathbb{C}\} \)
Proposition

If \( T \in \mathcal{L}(V) \), and \( k \) is a nonnegative integer, then

\[
\text{null } T^k \subseteq \text{null } T^{k+1}.
\]

Proposition (8.5)

- Let \( T \in \mathcal{L}(V) \), and
- suppose \( m \) is a nonnegative integer, such that
- \( \text{null } T^m = \text{null } T^{m+1} \).
- Then,

\[
\text{null } T^0 \subseteq \text{null } T^1 \subseteq \cdots \subseteq \text{null } T^m = \text{null } T^{m+1} = \cdots.
\]
Proposition (8.6)
If \( T \in \mathcal{L}(V) \), then

\[ \text{null } T^{\dim V} = \text{null } T^{\dim V+1} = \ldots. \]

Corollary (8.7)
- Suppose \( T \in \mathcal{L}(V) \), and \( \lambda \) is an eigenvalue of \( T \).
- Then the set of generalized eigenvalues of \( T \) corresponding to \( \lambda \) is

\[ \text{null}(T - \lambda I)^{\dim V}. \]
Nilpotent Operators

Definition
An operator is called \textit{nilpotent} if some power of it equals 0.

Example
- The map $N \in \mathcal{L}(\mathbb{F}^4)$ defined by $N(z_1, z_2, z_3, z_4) = (z_3, z_4, 0, 0)$ is nilpotent.
- The differentiation operator $D \in \mathcal{L}(\mathcal{P}_m(\mathbb{R}))$ is nilpotent.

Corollary (8.8)
If $N \in \mathcal{L}(V)$ is nilpotent, then $N^{\dim V} = 0$. 
Proposition
If $T \in \mathcal{L}(V)$, and $k$ is a nonnegative integer, then
\[ \operatorname{range} T^k \supseteq \operatorname{range} T^{k+1}. \]

Proposition (8.9)
If $T \in \mathcal{L}(V)$, then
\[ \operatorname{range} T^{\dim V} = \operatorname{range} T^{\dim V+1} = \ldots. \]
**Multiplicity of Eigenvalues**

**Theorem (8.10)**

- Let $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$.
- Then, for every basis of $V$ wrt. which $T$ has an upper-triangular matrix,
- $\lambda$ appears on the diagonal of the matrix of $T$ precisely $\dim \text{null}(T - \lambda I)^{\dim V}$ times.

**Definition**

- Suppose $\lambda$ is an eigenvalue of $T \in \mathcal{L}(V)$.
- The *multiplicity* of $\lambda$ is
- the dimension of the generalized eigenspace corresponding to $\lambda$,

$$\dim \text{null}(T - \lambda I)^{\dim V}.$$
Example

- Let $T \in \mathcal{L}(\mathbb{F}^3)$ be defined by $T(z_1, z_2, z_3) = (0, z_1, 5z_3)$.
- 0 is an eval. of $T$ with multiplicity 2, and
- 5 is an eval. of $T$ with multiplicity 1.

Example

- Suppose $T \in \mathcal{L}(\mathbb{F}^3)$ is the operator whose matrix wrt. the standard basis is
  \[
  \begin{pmatrix}
  6 & 7 & 7 \\
  0 & 6 & 7 \\
  0 & 0 & 7 \\
  \end{pmatrix}.
  \]
- Then 6 is an eval. of $T$ with multiplicity 2, and
- 7 is an eval. of $T$ with multiplicity 1.
Proposition (8.18)

1. If $V$ is a complex vector space, and $T \in \mathcal{L}(V)$,
2. then the sum of the multiplicities of all eigenvalues of $T$ equals $\dim V$. 
The Characteristic Polynomial of an Operator

Definition

Suppose $V$ is a complex vector space, and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eval.'s of $T$, and let $d_j$ denote the multiplicity of $\lambda_j$, for each $j$. The polynomial

$$(z - \lambda_1)^{d_1} \cdots (z - \lambda_m)^{d_m}$$

is called the characteristic polynomial of $T$.

The degree of the characteristic polynomial is $\dim V$.
The roots of the characteristic polynomial are the eval.'s of the operator.
Example

- Let $T \in \mathcal{L}(\mathbb{F}^3)$ be defined by $T(z_1, z_2, z_3) = (0, z_1, 5z_3)$.
- 0 is an eval. of $T$ with multiplicity 2, and
- 5 is an eval. of $T$ with multiplicity 1.
- Characteristic poly: $z^2(z - 5)$

Example

- Suppose $T \in \mathcal{L}(\mathbb{F}^3)$ is the operator whose matrix wrt. the standard basis is
  \[
  \begin{pmatrix}
  6 & 7 & 7 \\
  0 & 6 & 7 \\
  0 & 0 & 7
  \end{pmatrix}.
  \]
- Then 6 is an eval. of $T$ with multiplicity 2, and
- 7 is an eval. of $T$ with multiplicity 1.
- Characteristic poly: $(z - 6)^2(z - 7)$
Theorem (8.20)

- Suppose that $V$ is a complex vector space, and $T \in \mathcal{L}(V)$.
- If $q$ is the characteristic poly of $T$, then $q(T) = 0$. 
1. Generalized Eigenvectors
2. The Characteristic Polynomial
3. Decomposition of an Operator
4. Square Roots
5. The Minimal Polynomial
6. Jordan Form
Goal: For any \( T \in \mathcal{L}(V) \), show that \( V \) can be decomposed into the generalized eigenspaces of \( T \).
Invariance of $\text{null} \rho(T)$

**Proposition (8.22)**

- If $T \in \mathcal{L}(V)$, and
- $\rho \in \mathcal{P}(\mathbb{F})$,
- then $\text{null} \rho(T)$ is invariant under $T$. 
Decomposition of $V$ into Generalized Eigenspaces

**Theorem (8.23)**

- Suppose $V$ is a complex vector space, and $T \in \mathcal{L}(V)$.
- Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of $T$, and let $U_1, \ldots, U_m$ be the generalized eigenspaces.
- Then
  1. $V = U_1 \oplus \cdots \oplus U_m$;
  2. each $U_j$ is invariant under $T$;
  3. each $(T - \lambda_j I)|_{U_j}$ is nilpotent.

**Corollary (8.25)**

- Suppose $V$ is a complex vector space, and $T \in \mathcal{L}(V)$.
- Then there is a basis of $V$ consisting of generalized eigenvectors of $T$. 

(Tarleton State University) Math 550 Chapter 8 Fall 2010 20 / 36
Lemma (8.26)

- Suppose $N$ is a nilpotent operator on $V$.
- Then there is a basis of $V$ wrt. which the matrix of $N$ has the form
  \[
  \begin{pmatrix}
  0 & * \\
  \vdots & \ddots \\
  0 & 0 & 0
  \end{pmatrix}.
  \]
- All entries on and below the diagonal are zero.
Theorem (8.28)

Suppose $V$ is a complex vector space, and $T \in \mathcal{L}(V)$.

Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of $T$.

Then there is a basis of $V$ wrt. which $T$ has a block diagonal matrix of the form

$$
\begin{pmatrix}
A_1 & 0 \\
\vdots & \ddots \\
0 & \ldots & A_m
\end{pmatrix},
$$

where each $A_j$ is an upper-triangular matrix of the form

$$
\begin{pmatrix}
\lambda_j & * \\
\vdots & \ddots \\
0 & \ldots & \lambda_j
\end{pmatrix}.
$$
Outline

1. Generalized Eigenvectors
2. The Characteristic Polynomial
3. Decomposition of an Operator
4. Square Roots
5. The Minimal Polynomial
6. Jordan Form
Goal: Show that every *invertible* operator on a complex vector space has a square root.

**Example**

The operator $T \in \mathcal{L}(\mathbb{C}^3)$ defined by $T(z_1, z_2, z_3) = (z_2, z_3, 0)$ has no square root.
Lemma (8.30)

- Suppose $N \in \mathcal{L}(V)$ is nilpotent.
- Then $I + N$ has a square root.
Theorem (8.32)

- Suppose $V$ is a complex vector space.
- If $T \in \mathcal{L}(V)$ is invertible, then $T$ has a square root.

Invertible maps on complex vector spaces have $k$th roots, for any positive integer $k$. 
Outline

1. Generalized Eigenvectors
2. The Characteristic Polynomial
3. Decomposition of an Operator
4. Square Roots
5. The Minimal Polynomial
6. Jordan Form
The Minimal Polynomial

Definition

- Suppose \( p \in \mathcal{P}(F) \) is defined by
  \[
p(z) = a_0 + a_1 z + \cdots + a_m z^m.
  \]

- Then \( p \) is called a monic polynomial if \( a_m = 1 \).

Proposition

- Suppose \( T \in \mathcal{L}(V) \).
- There exists a unique monic polynomial \( p \) of smallest degree, such that \( p(T) = 0 \).
- This polynomial is called the minimal polynomial of \( T \).
Example

- The minimal polynomial of $I$ is $z - 1$.
- The minimal polynomial of
  \[
  \begin{pmatrix}
  4 & 1 \\
  0 & 5
  \end{pmatrix}
  \]
  is $20 - 9z + z^2$.

Clearly,

\[\text{deg min poly} \leq (\text{dim } V)^2.\]

By the Cayley-Hamilton Theorem on complex spaces,

\[\text{deg min poly} \leq \text{dim } V.\]

This holds on real spaces also (see Chapter 9).
Definition

- Suppose $p, q \in \mathcal{P}(\mathbb{F})$.
- Then $p$ divides $q$ if
- there exists some $s \in \mathcal{P}(\mathbb{F})$, such that $q = sp$.

Example

$(1 + 3z)^2$ divides $5 + 32z + 57z^2 + 18z^3$, because

$$5 + 32z + 57z^2 + 18z^3 = (2z + 5)(1 + 3z)^2.$$\

Theorem (8.34)

- Let $T \in \mathcal{L}(V)$, and $q \in \mathcal{P}(\mathbb{F})$.
- Then $q(T) = 0$ iff the minimal polynomial of $T$ divides $q$. 
Roots of the Minimal Polynomial are Precisely the Eigenvalues

Theorem (8.36)

- Suppose $T \in \mathcal{L}(V)$.
- Then the roots of the minimal polynomial of $T$ are precisely the eigenvalues of $T$. 
Calculating a Minimal Polynomial

- Find the smallest \( m \) such that

\[
M(I), M(T), \ldots, M(T)^m
\]

is linearly dependent.

- Then find scalars \( a_0, \ldots, a_{m-1} \) such that

\[
a_0 M(I) + \cdots + a_{m-1} M(T)^{m-1} + M(T)^m = 0.
\]

- These scalars are the coefficients of the minimal polynomial.
Example

Consider the operator on $\mathbb{C}^5$ whose matrix is

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & -3 \\
1 & 0 & 0 & 0 & 6 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}.
$$

The minimal polynomial is $z^5 - 6z + 3$.

The eigenvalues are $-1.67$, $0.51$, $1.40$, and $-0.12 \pm 1.59i$. 

Outline

1. Generalized Eigenvectors
2. The Characteristic Polynomial
3. Decomposition of an Operator
4. Square Roots
5. The Minimal Polynomial
6. Jordan Form
Suppose $T \in \mathcal{L}(V)$.

A basis of $V$ is called a Jordan Basis for $T$ if

the matrix of $T$ wrt. this basis is

$$
\begin{pmatrix}
A_1 & 0 \\
\vdots & \ddots \\
0 & A_m
\end{pmatrix},
$$

where each $A_j$ has the form

$$
A_j = 
\begin{pmatrix}
\lambda_j & 1 & 0 \\
\vdots & \ddots & \ddots \\
0 & \ddots & 1 \\
0 & \cdots & \lambda_j
\end{pmatrix}.
$$
Theorem (8.47)

- Suppose $V$ is a complex vector space, and $T \in \mathcal{L}(V)$.
- Then there is a Jordan basis for $T$.

See Chapter 12 of *Abstract Algebra*, by Dummit and Foote, for more info.