Math 5364 Homework 11

Do p. 105 (1, 9, 15) from Advanced Calculus by Folland.

1. Let $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^m$, and $y \in \mathbb{R}^n$. Show that
   \[
   \frac{\partial}{\partial x} x' Ay = Ay
   \]

2. If $\Sigma \in \mathbb{R}^{n \times n}$ is symmetric, show that
   \[
   \frac{\partial}{\partial x} x' \Sigma x = 2\Sigma x
   \]

3. Let $X \in \mathbb{R}^{n \times p}$ be a full rank matrix, and let $Y \in \mathbb{R}^n$. Find the vector $\gamma \in \mathbb{R}^p$ that minimizes $\|Y - X\gamma\|^2$. (Note that this is not a Lagrange multipliers problem, because there is no constraint on $\gamma$.)

4. Bonus: Let $\Sigma_{11} \in \mathbb{R}^{p_1 \times p_1}$ and $\Sigma_{22} \in \mathbb{R}^{p_2 \times p_2}$ be positive definite matrices. Also, let $\Sigma_{12} \in \mathbb{R}^{p_1 \times p_2}$ and define $\Sigma_{21} = \Sigma'_{12}$. If
   \[
   c_1 = \max \left\{ x' \Sigma_{12} y \mid x' \Sigma_{11} x = 1, y' \Sigma_{22} y = 1 \right\},
   \]
   is attained at $(x, y)$, show that
   \[
   \begin{pmatrix}
   -c_1 \Sigma_{11} & \Sigma_{12} \\
   \Sigma_{21} & -c_1 \Sigma_{22}
   \end{pmatrix}
   \begin{pmatrix}
   x \\
   y
   \end{pmatrix}
   = 0.
   \]

This is the main result from the theory of canonical correlations. This maximum is

\[
    c_1 = \max \{ \text{cov}(x'U, y'V) \mid \text{Var}(x'U) = 1, \text{Var}(y'V) = 1 \},
\]

where $U$ and $V$ are random vectors with joint covariance matrix

\[
\text{cov} \left[ \begin{pmatrix}
    U \\
    V
\end{pmatrix} \right] = \begin{pmatrix}
    \Sigma_{11} & \Sigma_{12} \\
    \Sigma_{21} & \Sigma_{22}
\end{pmatrix}.
\]