Math 505 Notes
Chapter 3

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3.1: The Binomial and Related Distributions

Section 3.2: The Poisson Distribution

Section 3.3: The Exponential, $\Gamma$, $\chi^2$, and $\beta$ Distributions

Section 3.4: The Normal Distribution

The Multivariate Normal Distribution

Section 4.1: Expectations of Functions
Definition
Let \( p \in [0, 1] \). A Bernoulli random variable with parameter \( p \) is a random variable \( X \) such that
- \( P(X = 1) = p \), and
- \( P(X = 0) = 1 - p \).

Definition
- Let \( n \in \mathbb{N}_+ \) and \( p \in [0, 1] \),
- and let \( X_1, \ldots, X_n \) be independent Bernoulli random variables with parameter \( p \).
- The random variable
  \[
  X = \sum_{i=1}^{n} X_i
  \]
  is a binomial random variable with parameters \( n \) and \( p \),
- denoted \( X \sim b(n, p) \).
Characteristics of a Binomial Random Variable

- Let $X \sim b(n, p)$.
- The p.m.f. of $X$ is
  \[ P[X = x] = \binom{n}{x} p^x (1 - p)^{n-x}, \text{ for } x = 0, 1, \ldots, n. \]
- The m.g.f. is
  \[ M(t) = (1 - p + pe^t)^n, \text{ for } t \in \mathbb{R}. \]
- $E(X) = np$
- $\text{Var}(X) = np(1 - p)$
Theorem

Let $X_1, \ldots, X_m$ be independent random variables such that

$X_i \sim \text{b}(n_i, p)$, for $i = 1, \ldots, m$.

Then

$$\sum_{i=1}^{m} X_i \sim \text{b} \left( \sum_{i=1}^{m} n_i, p \right).$$

Theorem

Suppose $X_1, \ldots, X_m$ are independent random variables with m.g.f.’s $M_1, \ldots, M_m$. Then the moment generating function of $\sum_{i=1}^{m} X_i$ is given by

$$M(t) = \prod_{i=1}^{m} M_i(t).$$
Definition

- Consider a sequence of independent Bernoulli trials with $P(\text{Success}) = p$, and let $r \in \mathbb{N}_+$.
- Let $Y$ be the number of failures that occur before the $r$th success.
- The p.m.f. for $Y$ is

$$P[Y = y] = \binom{y + r - 1}{r - 1} p^r (1 - p)^y, \text{ for } y = 0, 1, \ldots$$

and $Y$ is said to have a *negative binomial* distribution.

- A negative binomial distribution with $r = 1$ is called a *geometric distribution*.
Definition

- Consider a set of $N$ objects consisting of $N_1$ red objects and $N - N_1$ blue objects.
- Select $n$ of these objects at random, without replacement, and let $X$ be the number of red objects in the sample.
- Then the p.m.f. of $X$ is

$$P[X = x] = \frac{\binom{N_1}{x} \binom{N-N_1}{n-x}}{\binom{N}{n}},$$

and $X$ is said to have a hypergeometric distribution.
Outline

1. 3.1: The Binomial and Related Distributions
2. Section 3.2: The Poisson Distribution
3. Section 3.3: The Exponential, $\Gamma$, $\chi^2$, and $\beta$ Distributions
4. Section 3.4: The Normal Distribution
5. The Multivariate Normal Distribution
6. Section 4.1: Expectations of Functions
Poisson Process

- \( X \) has a *Poisson distribution* with parameter \( m \) if
  \[
P[X = x] = \frac{m^x e^{-m}}{x!}, \text{ for } x = 0, 1, \ldots
  \]

- Stream of “phone calls”
- Let \( g(x, w) \) be the probability of receiving \( x \) phone calls in a time interval of length \( w \)

**Assumptions:**
- \( g(1, h) \approx \lambda h \), for small \( h \)
- \( \sum_{x=2}^{\infty} g(x, h) \approx 0 \), for small \( h \)
- The number of phone calls in nonoverlapping intervals are independent.

\[
g(x, w) = \frac{(\lambda w)^x e^{-\lambda w}}{x!}
\]

- The number of phone calls received in an interval of length \( w \) is a Poisson random variable with parameter \( m = \lambda w \).
If $X$ has a Poisson distribution with parameter $m$,

- $M(t) = e^{m(e^t - 1)}$
- $E(X) = m$
- $\text{Var}(X) = m$

For a Poisson process, the parameter $\lambda$ represents the average number of “phone calls” in an interval of length 1.

The average number of “phone calls” in an interval of length $w$ is $\lambda w$.
A random variable $X$ has an *exponential distribution* with mean $\beta > 0$ if its p.d.f. is

$$f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}, \text{ for } x > 0.$$ 

The waiting time between “phone calls” in a Poisson process with parameter $\lambda$ is exponentially distributed with mean $\beta = \frac{1}{\lambda}$.

- Let $\alpha, \beta > 0$.
- 
  $$\Gamma(\alpha) := \int_{0}^{\infty} y^{\alpha-1} e^{-y} dy$$
- $\Gamma(\alpha) = (\alpha - 1)!$, for $\alpha \in \mathbb{N}_+$
- If the p.d.f. of $X$ is

  $$\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} \text{ for } x > 0,$$

then $X$ is said to have a *gamma distribution* with parameters $\alpha$ and $\beta$, i.e., $X \sim \Gamma(\alpha, \beta)$. 

(Tarleton State University) Chapter 3 Fall 2009 12 / 27
Let $\alpha \in \mathbb{N}_+$. The waiting time until the $\alpha$th “phone call” in a Poisson process with parameter $\lambda$ has the distribution $\Gamma(\alpha, \frac{1}{\lambda})$.

If $\alpha \in \mathbb{N}_+$, and $X_1, \ldots, X_\alpha$ are i.i.d. exponential random variables with parameter $\beta$, then

$$X_1 + \cdots + X_\alpha \sim \Gamma(\alpha, \beta).$$

$M(t) = (1 - \beta t)^{-\alpha}$, for $t < \beta^{-1}$.

$E(X) = \alpha \beta$

$\text{Var}(X) = \alpha \beta^2$

**Definition**

Let $\alpha = \frac{r}{2}$, where $r \in \mathbb{N}_+$, and $\beta = 2$.

The corresponding gamma distribution is called a $\chi^2$ distribution with $r$ degrees of freedom, denoted $\chi^2(r)$. 
Theorem

Let $X_1, \ldots, X_n$ be independent random variables.

- If $X_i \sim \Gamma(\alpha_i, \beta)$, for $i = 1, \ldots, n$, then $\sum_{i=1}^{n} X_i \sim \Gamma(\sum_{i=1}^{n} \alpha_i, \beta)$.
- If $X_i \sim \chi^2(r_i)$, for $i = 1, \ldots, n$, then $\sum_{i=1}^{n} X_i \sim \chi^2(\sum_{i=1}^{n} r_i)$.

Definition

Let $\alpha, \beta > 0$, and suppose the p.d.f. of $X$ is

$$f(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1 - x)^{\beta-1}, \text{ for } 0 < x < 1.$$ 

Then $X$ is said to have a beta distribution with parameters $\alpha$ and $\beta$. 
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Definition

A random variable $Z$ with pdf

$$f(z) = \exp \left\{-\frac{z^2}{2}\right\}, \; z \in \mathbb{R}$$

has a *standard normal distribution*.

- Its mean is 0.
- Its variance is 1.
Definition

A random variable $X$ has a normal distribution with mean $\mu$ and variance $\sigma^2$ if

$$X = \sigma Z + \mu,$$

where $Z$ is a standard normal random variable, i.e., if

$$Z = \frac{X - \mu}{\sigma}$$

has a standard normal distribution.

- $X \sim N(\mu, \sigma^2)$
- Its density is

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\}, \quad x \in \mathbb{R}.$$
Theorem

If \( X \sim N(\mu, \sigma^2) \), then

\[
V = \frac{(X - \mu)^2}{\sigma^2}
\]

has a \( \chi^2(1) \) distribution.

Theorem

Let \( X_1, \ldots, X_n \) be independent random variables such that \( X_i \sim N(\mu_i, \sigma_i^2) \), for \( i = 1, \ldots, n \). Then, given constants \( a_1, \ldots, a_n \),

\[
Y = \sum_{i=1}^{n} a_i X_i \sim N \left( \sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2 \right).
\]
Corollary

Let \( X_1, \ldots, X_n \) be i.i.d. normally distributed random variables with mean \( \mu \) and variance \( \sigma^2 \). Then

\[
\bar{X} = n^{-1} \sum_{i=1}^{n} X_i \sim N(\mu, \sigma^2 / n).
\]
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Definition

Let $Z_1, \ldots, Z_n$ be i.i.d. $N(0,1)$ random variables. Then the random vector $Z = (Z_1, \ldots, Z_n)'$ has a multivariate normal distribution with

- mean vector $E(Z) = 0$, and
- covariance matrix $\text{cov}(Z) = I_n$.
- $Z \sim N_n(0, I_n)$

$$M_Z(t) = \exp \left\{ \frac{1}{2} t' t \right\}.$$ 

$$f_Z(z) = \frac{1}{(2\pi)^{n/2}} \exp \left\{ -\frac{1}{2} z' z \right\}, \quad z \in \mathbb{R}^n.$$
The random vector $X$ has a multivariate normal distribution if

$$X = \Sigma^{1/2} Z + \mu,$$

where $\mu \in \mathbb{R}^n$, $\Sigma$ is an $n \times n$ positive semi-definite matrix, and $Z \sim \mathcal{N}_n(0, I_n)$.

- $E(X) = \mu$
- $\text{cov}(X) = \Sigma$
- $X \sim \mathcal{N}_n(\mu, \Sigma)$

$$M_X(t) = \exp \left\{ t' \mu + \frac{1}{2} t' \Sigma t \right\}$$

If $\Sigma$ is positive definite,

$$f_X(x) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp \left\{ -\frac{1}{2} (x - \mu)' \Sigma^{-1} (x - \mu) \right\}, \; x \in \mathbb{R}^n$$
Theorem

Suppose $X \sim N_n(\mu, \Sigma)$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$. Then

$$AX + b \sim N_m(A\mu + b, A\Sigma A').$$

Corollary

Suppose $X \sim N_n(\mu, \Sigma)$, and let $X_1$ and $X_2$ be be random vectors containing the first $n_1$ and last $n_2 = n - n_1$ components of $X$, respectively, i.e.,

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$ 

Partitioning $\mu$ and $\Sigma$ accordingly, we have

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \text{ and } \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$ 

Then $X_1 \sim N_{n_1}(\mu_1, \Sigma_{11})$, and $X_2 \sim N_{n_2}(\mu_2, \Sigma_{22})$. 

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Theorem

Let \( X = (X'_1, X'_2)' \), as in the previous corollary. Then \( X_1 \) and \( X_2 \) are independent if and only if \( \Sigma_{12} = 0 \). That is, two jointly normal random vectors (variables) are independent if and only if they are uncorrelated.

Theorem

Suppose \( X = (X'_1, X'_2)' \), as in the previous corollary. Then the conditional distribution of \( X_1 \) given \( X_2 \) is

\[
N_m(\mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (X_2 - \mu_2), \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} ).
\]
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Theorem

\[ E \left( \sum_{i=1}^{n} a_i X_i \right) = \sum_{i=1}^{n} a_i E(X_i) \]