Inverses of Relations and Functions

**Definition:** If \( f \) is a function then the *inverse of \( f \)*, written \( f^{-1} \), is the function obtained by "reversing" the rule of function \( f \).

For instance, if \( f(x) = x + 5 \) then \( f \) takes an input \( x \) and adds 5 to it to produce an output. To "undo" this, we must subtract 5: \( f^{-1}(x) = x - 5 \).

Similarly, the doubling function \( f(x) = 2x \)

is reversed by the "halving" function \( f^{-1}(x) = \frac{x}{2} \)

In general, a function \( g \) is the inverse of a function \( f \) if

\[
g(f(x)) = x \quad \text{for all } x \text{ in the domain of } f
g \text{ and } f(g(x)) = x \quad \text{for all } x \text{ in the domain of } g
\]

**Example**

Suppose we have a function \( f \) defined by \( f(x) = 2x - 5 \)

and we wish to find its inverse \( f^{-1}(x) \). First, replace \( f(x) \) with another letter. We will use \( y \):

\[
y = 2x - 5
\]

Solve for \( x \) in terms of \( y \):

\[
y = 2x - 5
y + 5 = 2x
\frac{y + 5}{2} = x
\]

This function has \( x \) as a function of \( y \), that is, whenever \( y \) is input, the output is \( \frac{y + 5}{2} \). This function is the inverse of function \( f \), written \( f^{-1} \). We will change the letter of the input variable to \( x \) and write

\[
f^{-1}(x) = \frac{x + 5}{2}
\]

Note that the graph of \( f \) and \( f^{-1} \) are symmetric about the line \( y = x \):
If $f^{-1}$ is the inverse function of $f$ then the composition of $f$ with $f^{-1}$ is the identity function:
\[ f \circ f^{-1}(x) = f(f^{-1}(x)) = x \]
and
\[ f^{-1} \circ f(x) = f^{-1}(f(x)) = x \]

**Example** Using $f(x) = 2x - 5$ and $f^{-1}(x) = \frac{x + 5}{2}$ we have
\[
\begin{align*}
    f(f^{-1}(x)) &= f\left(\frac{x + 5}{2}\right) \\
                 &= 2\left(\frac{x + 5}{2}\right) - 5 \\
                 &= x + 5 - 5 \\
                 &= x.
\end{align*}
\]
It is also the case in this example that $f^{-1}(f(x)) = x$, and you should check this.