

Mathematics through problem Solving

by Margaret Taplin
Institute of Sathya Sai Education, Hong Kong

What Is A 'Problem-Solving Approach'? As the emphasis has shifted from teaching problem solving to teaching *via* problem solving (Lester, Masingila, Mau, Lambdin, dos Santon and Raymond, 1994), many writers have attempted to clarify what is meant by a problem-solving approach to teaching mathematics. The focus is on teaching mathematical topics through problem-solving contexts and enquiry-oriented environments which are characterized by the teacher 'helping students construct a deep understanding of mathematical ideas and processes by engaging them in doing mathematics: creating, conjecturing, exploring, testing, and verifying' (Lester et al., 1994, p.154). Specific characteristics of a problem-solving approach include:

- interactions between students/students and teacher/students (Van Zoest et al., 1994)
- mathematical dialogue and consensus between students (Van Zoest et al., 1994)
- teachers providing just enough information to establish background/intent of the problem, and students clarifying, interpreting, and attempting to construct one or more solution processes (Cobb et al., 1991)
- teachers accepting right/wrong answers in a non-evaluative way (Cobb et al., 1991)
- teachers guiding, coaching, asking insightful questions and sharing in the process of solving problems (Lester et al., 1994)
- teachers knowing when it is appropriate to intervene, and when to step back and let the pupils make their own way (Lester et al., 1994)
- A further characteristic is that a problem-solving approach can be used to encourage students to make generalisations about rules and concepts, a process which is central to mathematics (Evan and Lappin, 1994).

Schoenfeld (in Olkin and Schoenfeld, 1994, p.43) described the way in which the use of problem solving in his teaching has changed since the 1970s:

My early problem-solving courses focused on problems amenable to solutions by Polya-type heuristics: draw a diagram, examine special

cases or analogies, specialize, generalize, and so on. Over the years the courses evolved to the point where they focused less on heuristics per se and more on introducing students to fundamental ideas: the importance of mathematical reasoning and proof..., for example, and of sustained mathematical investigations (where my problems served as starting points for serious explorations, rather than tasks to be completed).

Schoenfeld also suggested that a good problem should be one which can be extended to lead to mathematical explorations and generalisations. He described three characteristics of mathematical thinking:

1. valuing the processes of mathematization and abstraction and having the predilection to apply them
2. developing competence with the tools of the trade and using those tools in the service of the goal of understanding structure - mathematical sense-making (Schoenfeld, 1994, p.60).
3. As Cobb et al. (1991) suggested, the purpose for engaging in problem solving is not just to solve specific problems, but to 'encourage the interiorization and reorganization of the involved schemes as a result of the activity' (p.187). Not only does this approach develop students' confidence in their own ability to think mathematically (Schifter and Fosnot, 1993), it is a vehicle for students to construct, evaluate and refine their own theories about mathematics and the theories of others (NCTM, 1989). Because it has become so predominant a requirement of teaching, it is important to consider the processes themselves in more detail.

The Role of Problem Solving in Teaching Mathematics as a Process Problem solving is an important component of mathematics education because it is the single vehicle which seems to be able to achieve at school level all three of the values of mathematics listed at the outset of this article: functional, logical and aesthetic. Let us consider how problem solving is a useful medium for each of these. It has already been pointed out that mathematics is an essential discipline because of its practical role to the individual and society. Through a problem-solving approach, this aspect of mathematics can be developed. Presenting a problem and developing the skills needed to solve that problem is more motivational than teaching the skills without a context. Such motivation gives problem solving special value as a vehicle for learning new concepts and skills or the reinforcement of skills already acquired (Stanic and Kilpatrick, 1989, NCTM, 1989).

Approaching mathematics through problem solving can create a context which simulates real life and therefore justifies the mathematics rather than treating it as an end in itself. The National Council of Teachers of Mathematics (NCTM, 1980) recommended that problem solving be the focus of mathematics teaching because, they say, it encompasses skills and functions which are an important part of everyday life. Furthermore it can help people to adapt to changes and unexpected problems in their careers and other aspects of their lives. More recently the Council endorsed this recommendation (NCTM, 1989) with the statement that problem solving should underlie all aspects of mathematics teaching in order to give students experience of the power of mathematics in the world around them. They see problem solving as a vehicle for students to construct, evaluate and refine their own theories about mathematics and the theories of others. According to Resnick (1987) a problem-solving approach contributes to the practical use of mathematics by helping people to develop the facility to be adaptable when, for instance, technology breaks down. It can thus also help people to transfer into new work environments at this time when most are likely to be faced with several career changes during a working lifetime (NCTM, 1989). Resnick expressed the belief that 'school should focus its efforts on preparing people to be good adaptive learners, so that they can perform effectively when situations are unpredictable and task demands change' (p.18). Cockcroft (1982) also advocated problem solving as a means of developing mathematical thinking as a tool for daily living, saying that problem-solving ability lies 'at the heart of mathematics' (p.73) because it is the means by which mathematics can be applied to a variety of unfamiliar situations. Problem solving is, however, more than a vehicle for teaching and reinforcing mathematical knowledge and helping to meet everyday challenges. It is also a skill which can enhance logical reasoning. Individuals can no longer function optimally in society by just knowing the rules to follow to obtain a correct answer. They also need to be able to decide through a process of logical deduction what algorithm, if any, a situation requires, and sometimes need to be able to develop their own rules in a situation where an algorithm cannot be directly applied. For these reasons problem solving can be developed as a valuable skill in itself, a way of thinking (NCTM, 1989), rather than just as the means to an end of finding the correct answer. Many writers have emphasized the importance of problem solving as a means of developing the logical thinking aspect of mathematics. 'If education fails to contribute to the development of the intelligence, it is obviously incomplete. Yet intelligence is essentially the ability to solve problems: everyday problems, personal problems ... '(Polya, 1980, p.1). Modern

definitions of intelligence (Gardner, 1985) talk about practical intelligence which enables 'the individual to resolve genuine problems or difficulties that he or she encounters' (p.60) and also encourages the individual to find or create problems 'thereby laying the groundwork for the acquisition of new knowledge' (p.85). As was pointed out earlier, standard mathematics, with the emphasis on the acquisition of knowledge, does not necessarily cater for these needs. Resnick (1987) described the discrepancies which exist between the algorithmic approaches taught in schools and the 'invented' strategies which most people use in the workforce in order to solve practical problems which do not always fit neatly into a taught algorithm. As she says, most people have developed 'rules of thumb' for calculating, for example, quantities, discounts or the amount of change they should give, and these rarely involve standard algorithms. Training in problem-solving techniques equips people more readily with the ability to adapt to such situations. A further reason why a problem-solving approach is valuable is as an aesthetic form. Problem solving allows the student to experience a range of emotions associated with various stages in the solution process. Mathematicians who successfully solve problems say that the experience of having done so contributes to an appreciation for the 'power and beauty of mathematics' (NCTM, 1989, p.77), the "joy of banging your head against a mathematical wall, and then discovering that there might be ways of either going around or over that wall" (Olkin and Schoenfeld, 1994, p.43). They also speak of the willingness or even desire to engage with a task for a length of time which causes the task to cease being a 'puzzle' and allows it to become a problem. However, although it is this engagement which initially motivates the solver to pursue a problem, it is still necessary for certain techniques to be available for the involvement to continue successfully. Hence more needs to be understood about what these techniques are and how they can best be made available. In the past decade it has been suggested that problem-solving techniques can be made available most effectively through making problem solving the focus of the mathematics curriculum. Although mathematical problems have traditionally been a part of the mathematics curriculum, it has been only comparatively recently that problem solving has come to be regarded as an important medium for teaching and learning mathematics (Stanic and Kilpatrick, 1989). In the past problem solving had a place in the mathematics classroom, but it was usually used in a token way as a starting point to obtain a single correct answer, usually by following a single 'correct' procedure. More recently, however, professional organizations such as the National Council of Teachers of Mathematics (NCTM, 1980 and 1989) have recommended that the

mathematics curriculum should be organized around problem solving, focusing on:

- (i) developing skills and the ability to apply these skills to unfamiliar situations
- (ii) gathering, organising, interpreting and communicating information
- (iii) formulating key questions, analyzing and conceptualizing problems, defining problems and goals, discovering patterns and similarities, seeking out appropriate data, experimenting, transferring skills and strategies to new situations
- (iv) developing curiosity, confidence and open-mindedness (NCTM, 1980, pp.2-3).

One of the aims of teaching through problem solving is to encourage students to refine and build onto their own processes over a period of time as their experiences allow them to discard some ideas and become aware of further possibilities (Carpenter, 1989). As well as developing knowledge, the students are also developing an understanding of when it is appropriate to use particular strategies. Through using this approach the emphasis is on making the students more responsible for their own learning rather than letting them feel that the algorithms they use are the inventions of some external and unknown 'expert'. There is considerable importance placed on exploratory activities, observation and discovery, and trial and error. Students need to develop their own theories, test them, test the theories of others, discard them if they are not consistent, and try something else (NCTM, 1989). Students can become even more involved in problem solving by formulating and solving their own problems, or by rewriting problems in their own words in order to facilitate understanding. It is of particular importance to note that they are encouraged to discuss the processes which they are undertaking, in order to improve understanding, gain new insights into the problem and communicate their ideas (Thompson, 1985, Stacey and Groves, 1985).

Conclusion It has been suggested in this chapter that there are many reasons why a problem-solving approach can contribute significantly to the outcomes of a mathematics education. Not only is it a vehicle for developing logical thinking, it can provide students with a context for learning mathematical knowledge, it can enhance transfer of skills to unfamiliar situations and it is an aesthetic form in itself. A problem-

solving approach can provide a vehicle for students to construct their own ideas about mathematics and to take responsibility for their own learning. There is little doubt that the mathematics program can be enhanced by the establishment of an environment in which students are exposed to teaching via problem solving, as opposed to more traditional models of teaching about problem solving. The challenge for teachers, at all levels, is to develop the process of mathematical thinking alongside the knowledge and to seek opportunities to present even routine mathematics tasks in problem-solving contexts.

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