# A Brief Introduction to Game Theory

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# Outline

1. Games of Perfect Information
2. Games without Perfect Information
3. Final Thoughts

# Games of Perfect Information

- All players know all important details of the game state at all times.
- Games **with** perfect information:

  chess, checkers, tic-tac-toe

- Games **without** perfect information:

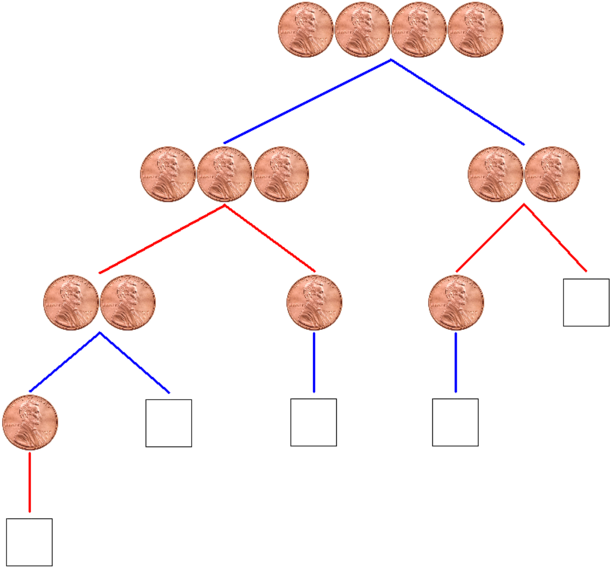
  poker, rock-paper-scissors

- Can be solved using **backwards induction.**

# Example of a Game with Perfect Information

- Two players.
- Start with 4 pennies in center of table.
- Each player can take 1 penny or 2 pennies on his/her turn.
- Player to take the last penny wins.
- First player = blue
- Second player = red

# Backwards Induction for Penny Game



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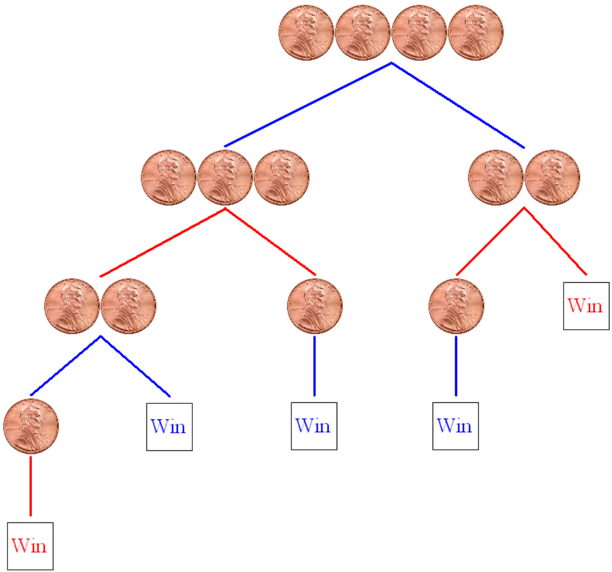

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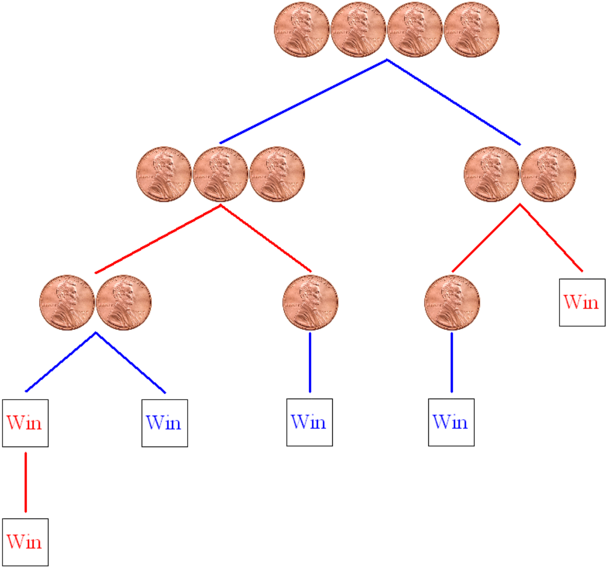

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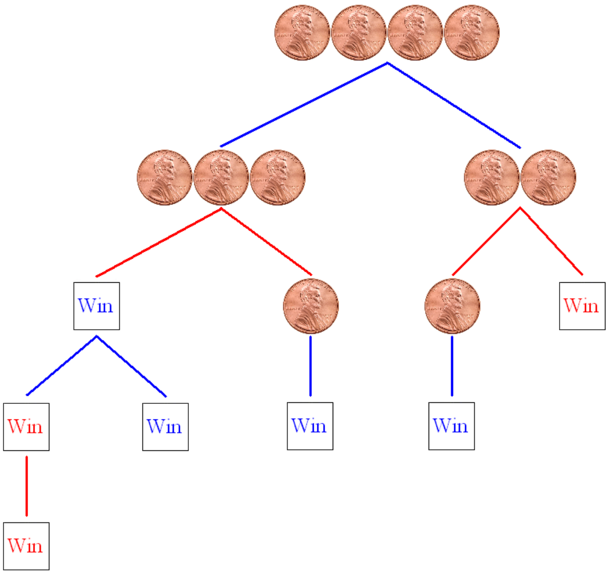

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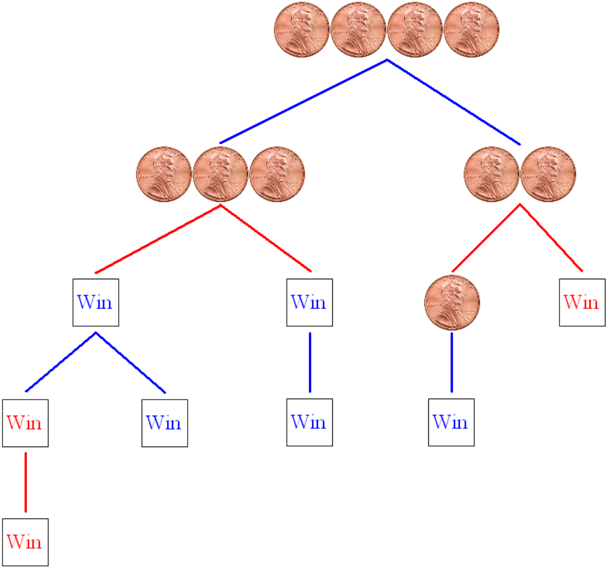

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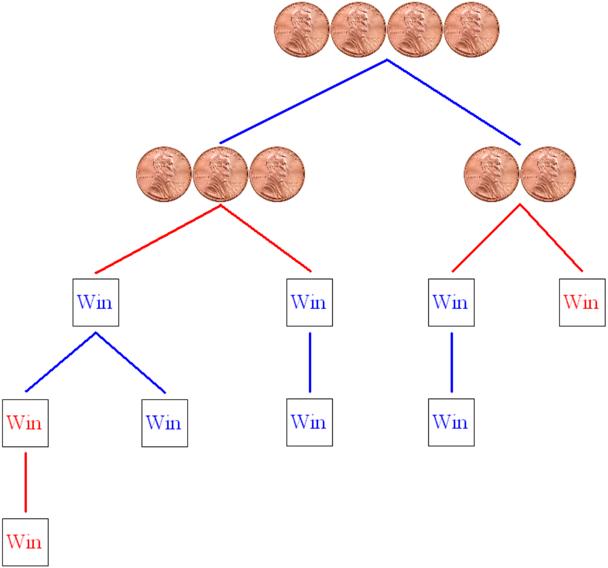

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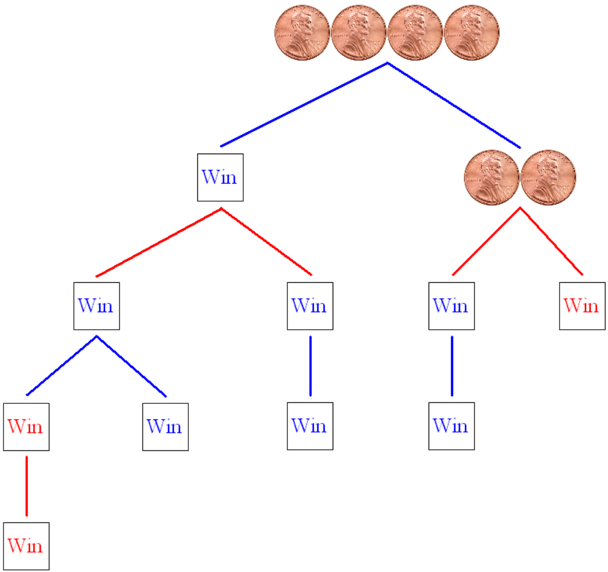

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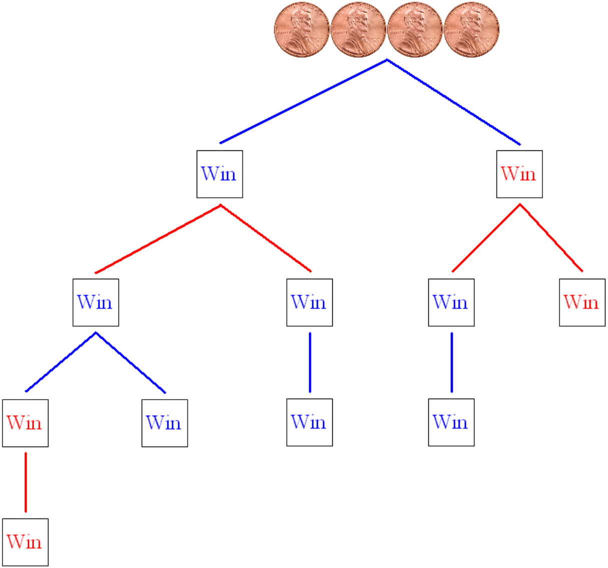

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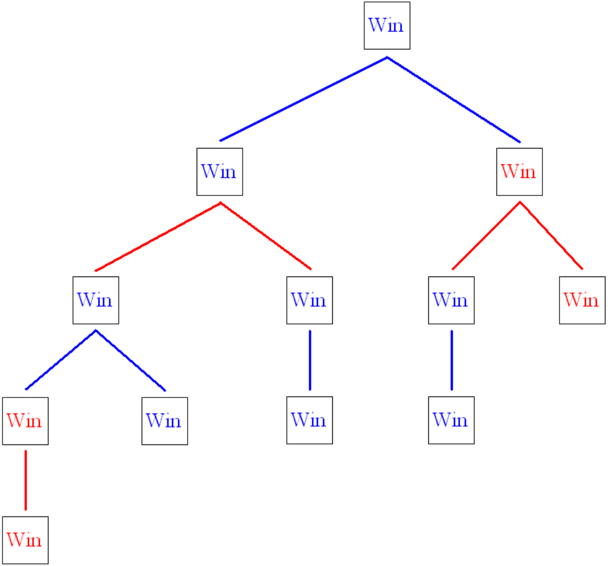
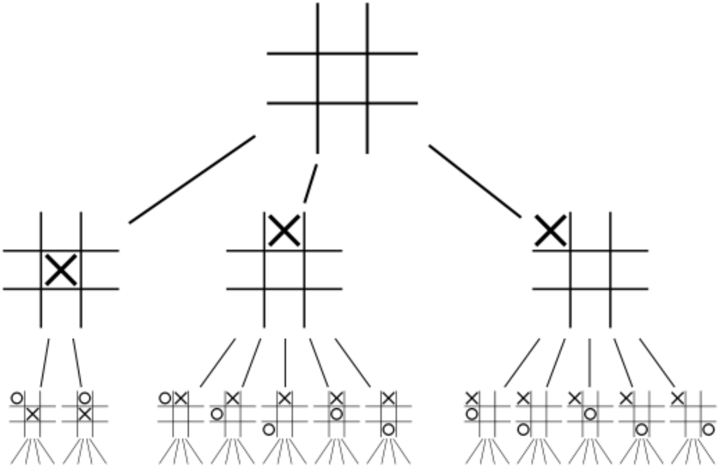


# Backwards Induction for Penny Game

- Conclusion: First player wins with optimal play.
- Backwards induction was easy.
- Number of variations $= 5$.

# Game Tree for Tic-Tac-Toe



# Tic-Tac-Toe and Checkers

- With optimal play, tic-tac-toe is a draw.
- Schaeffer et al. (2007) showed that checkers is also a draw.
- `http://webdocs.cs.ualberta.ca/~chinook/publications`

# Chess

- Number of variations is too big to use backwards induction.

  # of variations $> 14^{686} >$ Number of electrons in visible universe!

- Chess programs do use the game tree.
  - Only plot to finite depth.
  - Use an evaluation function to evaluate positions.
- `http://www.shredderchess.com/online-chess/online-da`

# Outline

# Rock-Paper-Scissors

- Two players
- Each one chooses Rock, Paper, or Scissors **simultaneously.**
- Rock beats Scissors
- Scissors beats Paper
- Paper beats Rock

# Payoff Matrix for RPS

- First player = blue
- Second player = red

| | Rock | Paper | Scissors |
|---|---|---|---|
| Rock | (0,0) | (-1,1) | (1,-1) |
| Paper | (1,-1) | (0,0) | (-1,1) |
| Scissors | (-1,1) | (1,-1) | (0,0) |

- RPS is a **zero sum game.**

# Randomized Strategy for RPS

- Need our strategy to be "snoop proof".
- Solution: use randomized strategy.
  - $p_R$ = probability of choosing Rock
  - $p_P$ = probability of choosing Paper
  - $p_S$ = probability of choosing Scissors
- Example:
  - $p_R = 0.7$
  - $p_P = 0.2$
  - $p_S = 0.1$

# Randomized Strategy for RPS

- Example:
  - $p_R = 0.7$
  - $p_P = 0.2$
  - $p_S = 0.1$
- If opponent chooses Paper, his expected utility is

$$0.7(1) + 0.2(0) + 0.1(-1) = 0.6$$

- This is the **maximum utility** our opponent can achieve.
- We want to **minimize** his maximum utility.

# Minimax Strategy for RPS

- Minimax strategy:
  - $p_R = \frac{1}{3}$
  - $p_P = \frac{1}{3}$
  - $p_S = \frac{1}{3}$
- Now if opponent chooses Paper, his expected utility is

$$\frac{1}{3}(1) + \frac{1}{3}(0) + \frac{1}{3}(-1) = 0$$

- No matter what he does, his expected utility will be 0.

# Equilibrium Strategies

- In zero sum games with finite strategy spaces, minimax strategies always exist for both players.
- Both players using minimax strategies is an **equilibrium**:
- Neither player can benefit from changing strategies.
- Theory can be generalized to multiplayer games, cf. Nash (1949).

# A Simplified Poker Game

- Both players ante $1.
- Player 1 is dealt a card that says "strong" or "weak".
  - 50% chance of getting "strong" card.
  - 50% chance of getting "weak" card.
- Player 1 may bet $1 or check.
- Player 2 may call or fold.
- If there's a showdown, Player 1 wins if card is strong and loses if card is weak.

- Player 1 should always bet with strong card.
- Questions:
  - How often should Player 1 bluff with weak card?
  - How often should Player 2 call when Player 1 bets?

# Expected Value for Player 1

- $p =$ probability that Player 1 bluffs with weak card
- $q =$ probability that Player 2 calls when Player 1 bets
- Player 1's expected value is

$$EV_1 = -1 + \frac{1}{2}[3q + 2(1-q)] + \frac{1}{2}p[-1q + 2(1-q)]$$

$$EV_1 = -1 + \frac{1}{2}[q + 2] + \frac{1}{2}p[2 - 3q]$$

# Optimal Calling Frequency

$$EV_1 = -1 + \frac{1}{2}[q+2] + \frac{1}{2}p[2-3q]$$

- Claim: Player 2 should choose $q = \frac{2}{3}$.
- If $q < \frac{2}{3}$, Player 1 can choose $p = 1$, and

$$EV_1 = 1 - q > \frac{1}{3}.$$

- If $q > \frac{2}{3}$, Player 1 can choose $p = 0$, and

$$EV_1 = \frac{1}{2}q > \frac{1}{3}.$$

- If $q = \frac{2}{3}$, then

$$EV_1 = \frac{1}{3}.$$

# A Bit of Algebra

$$EV_1 = -1 + \frac{1}{2}[q + 2] + \frac{1}{2}p[2 - 3q]$$

$$EV_1 = -1 + \frac{1}{2}[q(1 - 3p) + 2 + 2p]$$

# Optimal Bluffing Frequency

$$EV_1 = -1 + \frac{1}{2}[q(1 - 3p) + 2 + 2p]$$

- Claim: Player 1 should choose $p = \frac{1}{3}$.
- If $p < \frac{1}{3}$, Player 2 can choose $q = 0$, and
$$EV_1 = p < \frac{1}{3}.$$
- If $p > \frac{1}{3}$, Player 2 can choose $q = 1$, and
$$EV_1 = \frac{1}{2} - \frac{1}{2}p < \frac{1}{3}.$$
- If $p = \frac{1}{3}$, then
$$EV_1 = \frac{1}{3}.$$

# Simplified Poker Game Solution

- Player 1 should always bet with a strong card.
- Player 1 should bluff $\frac{1}{3}$ of the time with a weak card.
- Player 2 should call $\frac{2}{3}$ of the time when Player 1 bets.
- Player 1 will win about 33 cents per hand on average.

# Outline

# A Non-zero-sum Game: Prisoner's Dilemma

- Two criminals
- Interrogated in separate rooms

| | Stay Silent | Confess |
|---|---|---|
| Stay Silent | (-1,-1) | (-10,0) |
| Confess | (0,-10) | (-5,-5) |

- General principle: individuals acting in their own self interest can lead to a negative outcome for the group.
- Related problems:
  - Pollution/"Tragedy of the Commons"
  - Cartels/Monopolies
  - Taxation and public goods

# Areas of Application for Game Theory

- Economics/Political Science
  - Bargaining problems
- Biology
  - Competition between organisms
  - Sex ratios
  - Genetics
- Philosophy

# References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- Nash, J.F. (1949). Equilibrium Points in $N$-Person Games. *Proceedings of the National Academy of Sciences* **36** 48-49.
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Thank You!