A Brief Introduction to Game Theory

Jesse Crawford

Department of Mathematics
Tarleton State University

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The Penny Game

- Two players.
- Start with 4 pennies in center of table.
- Each player can take 1 penny or 2 pennies on his/her turn.
- Player to take the last penny wins.
- First player = blue
- Second player = red
Backwards Induction for Penny Game

- Wins are indicated by "Win" boxes.
- The tree structure represents the game's possible moves and outcomes.

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Backwards Induction for Penny Game
Backwards Induction for Penny Game

The diagram represents a decision tree for a penny game, with nodes labeled "Win". The tree structure indicates the sequence of decisions and outcomes in the game.
Backwards Induction for Penny Game
Backwards Induction for Penny Game

(Backwards Induction Diagram)

Win

Win

Win

Win

Win

Win

Win

Win

Win
Game Tree for Tic-Tac-Toe
With optimal play, tic-tac-toe is a draw.

Schaeffer et al. (2007) showed that checkers is also a draw.
First turn white moves = 20

First turn black moves = 20

(20)(20) = 400

400 variations on first two moves!
Allis (1994) estimated number of chess variations:

\[ # \text{ of variations} > 10^{123} > \text{Number of atoms in visible universe!} \]

Chess programs do use the game tree.
- Only plot to finite depth.
- Use an evaluation function to evaluate positions.
Ticket costs $1

<table>
<thead>
<tr>
<th>Prize</th>
<th>$0</th>
<th>$2</th>
<th>$10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.8</td>
<td>0.15</td>
<td>0.05</td>
</tr>
</tbody>
</table>
Scratch-off Lottery Ticket

Ticket costs $1

<table>
<thead>
<tr>
<th>x</th>
<th>$0</th>
<th>$2</th>
<th>$10</th>
</tr>
</thead>
<tbody>
<tr>
<td>p(x)</td>
<td>0.8</td>
<td>0.15</td>
<td>0.05</td>
</tr>
</tbody>
</table>

**Expected Value**

\[
E(X) = \sum xp(x) = 0 \cdot 0.8 + 2 \cdot 0.15 + 10 \cdot 0.05 = 0.8
\]
Pot size = 23 chips
2 chips to call

- Pot = 23
- Bet = 2
- Probability of hitting our straight = \( p = \frac{4}{46} = 0.087 \).
If we fold

<table>
<thead>
<tr>
<th>$x$</th>
<th>$0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x)$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

$E(X) = 1(0) = 0$

If we call

<table>
<thead>
<tr>
<th>$x$</th>
<th>$23$</th>
<th>$-2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x)$</td>
<td>$p$</td>
<td>$(1 - p)$</td>
</tr>
</tbody>
</table>

$E(X) = 23p - 2(1 - p)$

We should call if

$$23p - 2(1 - p) \geq 0$$

$$p \geq \frac{2}{23 + 2} = 0.08$$
Rock-Paper-Scissors

- Two players
- Each one chooses Rock, Paper, or Scissors **simultaneously**.
Payoff Matrix for RPS

- First player = blue
- Second player = red

<table>
<thead>
<tr>
<th></th>
<th>Rock</th>
<th>Paper</th>
<th>Scissors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rock</td>
<td>(0,0)</td>
<td>(-1,1)</td>
<td>(1,-1)</td>
</tr>
<tr>
<td>Paper</td>
<td>(1,-1)</td>
<td>(0,0 )</td>
<td>(-1,1)</td>
</tr>
<tr>
<td>Scissors</td>
<td>(-1,1)</td>
<td>(1,-1)</td>
<td>(0,0)</td>
</tr>
</tbody>
</table>

- **RPS is a zero sum game.**
Randomized Strategy for RPS

- Randomized strategy.
  - $p_R =$ probability of choosing Rock
  - $p_P =$ probability of choosing Paper
  - $p_S =$ probability of choosing Scissors

- Example:
  - $p_R = 0.7$
  - $p_P = 0.2$
  - $p_S = 0.1$
Randomized Strategy for RPS

Example:

- \( p_R = 0.7 \)
- \( p_P = 0.2 \)
- \( p_S = 0.1 \)

If opponent chooses Paper, his expected utility is

\[
0.7(1) + 0.2(0) + 0.1(-1) = 0.6
\]

This is the maximum utility our opponent can achieve.

We want to minimize his maximum utility.
Minimax Strategy for RPS

- Minimax strategy:
  - \( p_R = \frac{1}{3} \)
  - \( p_P = \frac{1}{3} \)
  - \( p_S = \frac{1}{3} \)

- Now if opponent chooses Paper, his expected utility is
  \[
  \frac{1}{3}(1) + \frac{1}{3}(0) + \frac{1}{3}(-1) = 0
  \]

- No matter what he does, his expected utility will be 0.
In zero sum games with finite strategy spaces, minimax strategies always exist for both players.

Theory can be generalized to multiplayer games, cf. Nash (1949).
King-Two-Jack Game

- Both players ante $1
- Player 1 may bet $1 or check
- Player 2 calls or folds
- Highest card wins
King-Two-Jack Game

- Both players ante $1
- Player 1 may bet $1 or check
- Player 2 calls or folds
- Highest card wins
- Player 1 always bets with King
- $p = \Pr(\text{Player 1 bluffs with 2})$
- $q = \Pr(\text{Player 2 calls})$
Expected Value for Player 1

- \( p = \) probability that Player 1 bluffs with 2♣.
- \( q = \) probability that Player 2 calls when Player 1 bets
- Player 1’s expected value is

\[
EV_1 = -1 + \frac{1}{2} [3q + 2(1 - q)] + \frac{1}{2} p[-1q + 2(1 - q)]
\]

\[
EV_1 = -1 + \frac{1}{2} [q + 2] + \frac{1}{2} p[2 - 3q]
\]
\[ EV_1 = -1 + \frac{1}{2} [q + 2] + \frac{1}{2} p [2 - 3q] \]

- **Claim:** Player 2 should choose \( q = \frac{2}{3} \).
- If \( q < \frac{2}{3} \), Player 1 can choose \( p = 1 \), and
  \[ EV_1 = 1 - q > \frac{1}{3}. \]
- If \( q > \frac{2}{3} \), Player 1 can choose \( p = 0 \), and
  \[ EV_1 = \frac{1}{2} q > \frac{1}{3}. \]
- If \( q = \frac{2}{3} \), then
  \[ EV_1 = \frac{1}{3}. \]
\[ EV_1 = -1 + \frac{1}{2} [q + 2] + \frac{1}{2} p [2 - 3q] \]

\[ EV_1 = -1 + \frac{1}{2} [q(1 - 3p) + 2 + 2p] \]
Optimal Bluffing Frequency

\[ EV_1 = -1 + \frac{1}{2}[q(1 - 3p) + 2 + 2p] \]

- Claim: Player 1 should choose \( p = \frac{1}{3} \).
- If \( p < \frac{1}{3} \), Player 2 can choose \( q = 0 \), and
\[ EV_1 = p < \frac{1}{3}. \]
- If \( p > \frac{1}{3} \), Player 2 can choose \( q = 1 \), and
\[ EV_1 = \frac{1}{2} - \frac{1}{2}p < \frac{1}{3}. \]
- If \( p = \frac{1}{3} \), then
\[ EV_1 = \frac{1}{3}. \]
King-Two-Jack Solution

- Player 1 should always bet with \(K\spadesuit\).
- Player 1 should bluff \(\frac{1}{3}\) of the time with \(2\spadesuit\).
- Player 2 should call \(\frac{2}{3}\) of the time when Player 1 bets.
- Player 1 will win about 33 cents per hand on average.
A Non-zero-sum Game: Prisoner’s Dilemma

- Two criminals
- Interrogated in separate rooms

<table>
<thead>
<tr>
<th></th>
<th>Stay Silent</th>
<th>Confess</th>
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<tbody>
<tr>
<td>Stay Silent</td>
<td>(-1,-1)</td>
<td>(-10,0)</td>
</tr>
<tr>
<td>Confess</td>
<td>(0,-10)</td>
<td>(-5,-5)</td>
</tr>
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General principle: individuals acting in their own self interest can lead to a negative outcome for the group.

Related problems:
- Pollution/“Tragedy of the Commons”
- Cartels/Monopolies
- Taxation and public goods
Areas of Application for Game Theory

- **Economics/Political Science**
  - Bargaining problems

- **Biology**
  - Competition between organisms
  - Sex ratios
  - Genetics

- **Philosophy**


Thank You!