Math 3320 Foundations of Mathematics
Chapter 1: Fundamentals

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Outline

1 Section 1.1: Why Study Foundations of Mathematics?
2 Section 1.2: Speaking (and Writing) Mathematics
3 Section 1.3: Definition
4 Section 1.4: Theorem
5 Appendix D
6 Section 1.5: Proof
7 Section 1.6: Counterexample
8 Section 1.7: Boolean Algebra
Is the following statement always true?

\[ \lim_{n \to \infty} \int_{a}^{b} f_n(x) \, dx = \int_{a}^{b} \lim_{n \to \infty} f_n(x) \, dx \]

Example

For \( n = 1, 2, 3, \ldots \), define

\[ f_n(x) = \begin{cases} 
  n, & \text{if } 0 \leq x \leq \frac{1}{n} \\
  0, & \text{otherwise.} 
\end{cases} \]

\[ \lim_{n \to \infty} \int_{0}^{1} f_n(x) \, dx \neq \int_{0}^{1} \lim_{n \to \infty} f_n(x) \, dx \]

The above example is a counterexample to the statement above.
Lebesgue’s Dominated Convergence Theorem

- Let \( \{f_n\} \) be a sequence of real-valued measurable functions on a measure space \((S, \Sigma, \mu)\).
- Suppose that the sequence converges pointwise to a function \( f \) and is dominated by some integrable function \( g \) in the sense that
  \[
  |f_n(x)| \leq g(x),
  \]
  for all numbers \( n \) in the index set and all points \( x \in S \).
- Then \( f \) is integrable, and
  \[
  \lim_{n \to \infty} \int_S f_n \, d\mu = \int_S \lim_{n \to \infty} f_n \, d\mu.
  \]

Goal of this Course

- Transform from a “symbol pushing” student to one who understands foundations of mathematics.
- You will be able to understand and prove theorems like this one!
Foundations Overview

- **Cornerstones of mathematics**: definition, theorem, and proof.
- **Mathematical concepts** must be precisely **defined**.
- **Theorems** are statements about these concepts.
  - $2 + 2 = 4$
  - $\frac{d}{dx} \sin(x) = \cos(x)$
  - “Two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.”
  - Lebesgue’s Dominated Convergence Theorem
- **Theorems** must be **proved** according to sound logic.
Set Theory and Logic

- **Set Theory:**
  - \( \{1, 2, 3\} \)
  - \( \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \)
  - \( \mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\} \)
  - \( \mathbb{R} = \{\text{All real numbers}\} \)
  - \( [a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\} \)

- **Logic**
  - Boolean operators: and, or, not, if-then, if and only if, implies.
  - Words used to structure proofs: Let, assume, suppose, therefore, then
  - Logical Quantifiers:
    - Universal: for all, for every, \( \forall \)
    - Existential: there exists, for some, \( \exists \)

**Example**

Compare these two statements:

- For all \( x \in \mathbb{R}, x^2 = 9. \)
- There exists \( x \in \mathbb{R}, \text{ such that } x^2 = 9. \)
Lebesgue’s Dominated Convergence Theorem

- Let \( \{f_n\} \) be a sequence of real-valued measurable functions on a measure space \((S, \Sigma, \mu)\).
- Suppose that the sequence converges pointwise to a function \( f \) and is dominated by some integrable function \( g \) in the sense that
  \[
  |f_n(x)| \leq g(x),
  \]
  for all numbers \( n \) in the index set and all points \( x \in S \).
- Then \( f \) is integrable, and
  \[
  \lim_{n \to \infty} \int_S f_n \, d\mu = \int_S \lim_{n \to \infty} f_n \, d\mu.
  \]
Example

What is \( \lim_\limits_{x \to 0} x \sin(\frac{1}{x}) \)?

Definition

Let \( f \) be a function defined on some interval containing \( a \), except possibly at \( a \) itself. Then we write

\[
\lim_\limits_{x \to a} f(x) = L,
\]

if, for every \( \varepsilon > 0 \), there exists \( \delta > 0 \), such that

\[
0 < |x - a| < \delta \text{ implies } |f(x) - L| < \varepsilon.
\]

Theorem

\[
\lim_\limits_{x \to 0} x \sin(\frac{1}{x}) = 0
\]
A Misconception About Proofs

- Misconception: Proofs are just about formatting the text with a specific style, because my teacher is picky!
- Reality:
  - As with any writing, it’s important to be professional, but the chosen presentation format is not that big of a deal.
  - However, changing a single word in a proof can be extremely important, because it can change the meaning of that sentence and cause the proof to be logically incorrect.
## Unique Features of Mathematics

- You don’t rely on experiments or third party accounts in math. You can prove/disprove things.
- You don’t have to take someone else’s word for it.
- You can obtain (close to) certain knowledge.
- There really is a right answer, and you can determine what it is.
- Mathematics is the foundation for all science and technology, so we need logically sound methods for deriving mathematical knowledge.
- Math is a fun, puzzle-solving activity.
Speaking and Writing Mathematics

- Precision is a top priority. We want to avoid being vague or unclear.

- Complete Sentences
  - **Bad:** $3x + 5$
  - **Good:** When we substitute $x = -5/3$ into $3x + 5$, the result is 0.

- Mismatch of Categories
  - “Air Force One is the President of the United States.”
  - **Bad:** “If the legs of a right triangle $T$ have lengths 5 and 12, then $T = 30$.”
  - **Good:** “If the legs of a right triangle $T$ have lengths 5 and 12, then the area of $T$ is 30.”

- Avoid Pronouns
  - **Bad:** “If we move everything over, then it simplifies and that’s our answer.”
  - **Good:** “When we move all terms involving $x$ to the left in Equation (12), we find those terms cancel, and that enables us to determine the value of $y$.”
Speaking and Writing Mathematics

- Rewrite your proofs
- Learn Latex
1.1: Why Study Foundations of Mathematics?

1.2: Speaking (and Writing) Mathematics

1.3: Definition

1.4: Theorem

Appendix D

1.5: Proof

1.6: Counterexample

1.7: Boolean Algebra
Definition (Even)

An integer is called *even* provided it is divisible by two.
Definition (Even)

An integer is called even provided it is divisible by two.

- For this definition to make sense, we need to define the terms in red.
- That would require us to define even more terms.

Eventually we hit the foundation: **Set Theory** (Chapter 2)
Our Starting Point: The Integers

\[ \mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \} \]

You may use the following freely in any proof:

- Algebraic properties of addition, subtraction, and multiplication (not division).
- Basic number facts like \( 3 \times 2 = 6 \).
- Basic facts about the order relations \((<, \leq, >, \geq)\).
- For specific details, see Appendix D.

Definition (Even)

An integer is called even provided it is divisible by two.
Definition (Divisible)

- Let $a$ and $b$ be integers.
- We say that $a$ is divisible by $b$ provided there is an integer $c$, such that $bc = a$.
- We also say that $b$ divides $a$, or $b$ is a factor of $a$, or $b$ is a divisor of $a$.
- The notation for this is $b|a$.

Definition (Even)

An integer is called even provided it is divisible by two.

Definition (Odd)

An integer $a$ is called odd provided there is an integer $x$, such that $a = 2x + 1$. 
Definition (Prime)
An integer $p$ is prime provided that $p > 1$ and the only positive divisors of $p$ are 1 and $p$.

Definition (Composite)
A positive integer $a$ is composite provided there is an integer $b$, such that $1 < b < a$, and $b | a$.

General Form of a Definition
An object $X$ is called the term being defined provided it satisfies specific conditions.
Homework

- **To Turn In:** p. 6 (1–7, 9, 12, 13a)
- **To Discuss:** p. 6 (1cefg, 2, 3ce, 4, 9, 12abc)
A *theorem* is a declarative statement about mathematics for which there is a proof.

- Declarative: Not a command, not a question.
- Theorems are true.

**The Pythagorean Theorem**

If $a$ and $b$ are the lengths of the legs of a right triangle, and $c$ is the length of the hypotenuse, then

$$a^2 + b^2 = c^2$$
Other Names for Theorems

- **Fact:** $6 + 3 = 9$
- **Proposition/Result:** A minor theorem
- **Lemma:** Theorem primarily used to prove another theorem
- **Corollary:** Theorem that follows immediately from another
- **Claim:** Theorem often used inside of the proof of another theorem
A False Statement
For any real number $x$,
\[
\sqrt{x^2} = x.
\]

A Nonsense Statement
The square root of a triangle is a circle.
A *conjecture* is a statement about mathematics whose truth is unknown.

**Goldbach’s Conjecture**

Every even integer greater than 2 can be expressed as the sum of two primes.
“If John mows my lawn, I will pay him $20.”

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>If $A$, then $B$.</th>
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</thead>
<tbody>
<tr>
<td>$T$</td>
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Other Names for “If $A$, then $B$.”

- $A$ implies $B$.
- $A \Rightarrow B$
- $B \iff A$
- Whenever $A$ is true, $B$ is true.
- $A$ is sufficient for $B$.
- $B$ is necessary for $A$. 
“If John mows my lawn, I will pay him $20.”

Truth Table for If-Then

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<th>If $A$, then $B$.</th>
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Example

- The sum of two even integers is even.
- All trucks are vehicles.
- All vehicles are trucks.
- All nonvehicles are nontrucks.
- All differentiable functions are continuous.
Contrapositive and Converse

Assume the statement, “If $A$, then $B$,” is true.

- **Contrapositive:** “If not $B$, then not $A$.”
  
The contrapositive is logically equivalent to the original statement, so it is also true.

- **Converse:** “If $B$, then $A$.”
  
The converse is not logically equivalent to the original statement, so it may or may not be true.
If $x$ is an even integer, then $x + 1$ is an odd integer.

If $x + 1$ is an odd integer, then $x$ is an even integer.

$x$ is an even integer if and only if $x + 1$ is an odd integer.

“I will pay John $20, if and only if he mows my lawn.”

Truth Table for If and Only If

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<th>$A$</th>
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<th>$A$ if and only if $B$.</th>
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</table>
If and Only If

Other Names for If and Only If

- $A$ iff $B$.
- $A \iff B$
- $A$ is necessary and sufficient for $B$.
- $A$ is equivalent to $B$. 
And

Truth Table for And

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<th>A and B.</th>
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</thead>
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Example

Which of these statements is true?

- $2 + 2 = 4$ and $\frac{d}{dx} \sin(x) = \cos(x)$
- $3 \mid 15$ and 1 is prime.
- 6 is an odd integer and 7 is a composite integer
- If $x$ and $y$ are integers such that $x^2 + y^2 = 0$, then $x = 0$ and $y = 0$. 
Truth Table for Or

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<th>$A$</th>
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<th>$A$ or $B$.</th>
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</tbody>
</table>

Example

Which of these statements is true?

- $2 + 2 = 4$ or $\frac{d}{dx} \sin(x) = \cos(x)$
- $3|15$ or $1$ is prime.
- $6$ is an odd integer or $7$ is a composite integer
- If $x$ and $y$ are integers such that $xy = 0$, then $x = 0$ or $y = 0$. 
### Truth Table for Not

<table>
<thead>
<tr>
<th>$A$</th>
<th>Not $A$</th>
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<tbody>
<tr>
<td>T</td>
<td>F</td>
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<td>F</td>
<td>T</td>
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</tbody>
</table>

### Example

Which of these statements is true?

- It is not the case that $\frac{d}{dx} e^x = \cos(x)$
- 4 does not divide 20.
Contrapositives and Converse Revisited

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$A \Rightarrow B$</th>
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</thead>
<tbody>
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<td>T</td>
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<td>T</td>
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<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

- **Contrapositive:** \((\text{not } B) \Rightarrow (\text{not } A)\)
- **Converse:** \(B \Rightarrow A\)

Example

- Construct truth tables for the contrapositive and converse.
- This approach will help with homework problems 4 and 6.
Vacuous Truth

Definition
- Consider the statement, “If $A$ then $B$”.
- If it is impossible for $A$ to be true, then the above statement is true.
- In this case, it is called *vacuously* true.

Example
- If an integer is both a perfect square and prime, then it is negative.
- If Santa Claus mows my lawn, I will pay him $1,000,000.
- All of my children are Nobel Prize winners.
- All of my children are convicted felons.
Homework

- **To Turn In:** p. 13 (1, 2, 4, 6, 7, 10, 12ace)
- **To Discuss:** p. 13 (1ab, 2ac, 4, 10, 12ce)
Outline

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8. Section 1.7: Boolean Algebra
Our Starting Point: The Integers

\[ \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \]

You may use the following freely in any proof:

- Algebraic properties of addition, subtraction, and multiplication (not division).
- Basic number facts like \(3 \times 2 = 6\).
- Basic facts about the order relations \((<, \leq, >, \geq)\).
- For specific details, see Appendix D.
Some Properties of the Integers

There exists a set \( \mathbb{Z} \) called the **integers**, and binary operations defined on \( \mathbb{Z} \) called **addition** and **multiplication** (denoted \( x + y \) and \( xy \)), satisfying the following conditions:

For any integers \( x, y, \) and \( z \),

- **Closure Property:** \( x + y \) and \( xy \) are also integers
- **Commutative Properties:** \( x + y = y + x \), and \( xy = yx \)
- **Associative Properties:** \( x + (y + z) = (x + y) + z \) and \( x(yz) = (xy)z \)
- **Distributive Property:** \( x(y + z) = xy + xz \)

**Additive Identity:** There exists an integer 0, such that \( x + 0 = x \), for any integer \( x \).

**Additive Inverse:** For any integer \( x \), there exists an integer \(-x\), such that \( x + (-x) = 0 \).

**Multiplicative Identity:** There exists an integer 1, such that \( 1x = x \), for any integer \( x \).
Some Properties of the Order Relations $<, >, \leq, \geq$

Let $a, b, c, \text{ and } d$ be integers.

- If $a < b$ and $c < d$, then $a + c < b + d$.
- Let $x$ be a positive integer. Then $a < b$ if and only if $ax < bx$.
- **Transitive Property:** If $a < b$, and $b < c$, then $a < c$.
- The above properties are all true for $>, \leq, \text{ and } \geq$ also.
Proposition 5.2

The sum of two even integers is even.

Proof Template 1: Direct Proof of an If Then statement.

To prove the statement, “If $A$, then $B$”

Assume $A$

: 

Make logical deductions

: 

Conclude $B$
Proposition 5.3
Let $a$, $b$, and $c$ be integers. If $a|b$ and $b|c$, then $a|c$.

What can we say about $x^3 + 1$ if $x$ is a positive integer? Prime or composite?

\[
\begin{align*}
1^3 + 1 &= 2 \\
2^3 + 1 &= 9 \\
3^3 + 1 &= 28 \\
4^3 + 1 &= 65 \\
5^3 + 1 &= 126
\end{align*}
\]

Proposition 5.4
Let $x$ be an integer. If $x > 1$, then $x^3 + 1$ is composite.
Proposition 5.5
Let \( x \) be an integer. Then \( x \) is even if and only if \( x + 1 \) is odd.

Proof Template 2: Direct Proof of an If and Only If Statement.
To prove the statement, “\( A \) iff \( B \)”

- \((\Rightarrow)\) Prove “If \( A \), then \( B \)”
- \((\Leftarrow)\) Prove “If \( B \), then \( A \)”
Homework

- **To Turn In:** p. 22 (1–3, 5, 7–9, 15, 20, 24)
- **To Discuss:** p. 22 (1, 3, 7, 15, 20, 24)
False Statement
Let $a$ and $b$ be integers. If $a | b$ and $b | a$, then $a = b$.

Proof Template 3: Refuting a False If-Then Statement with a Counterexample

- To disprove a statement of the form “If $A$, then $B$”
- Find an instance where $A$ is true but $B$ is false.

Example
Disprove: If $p$ and $q$ are prime, then $p + q$ is composite.
TRUE $\land$ TRUE = TRUE
TRUE $\land$ FALSE = FALSE
FALSE $\land$ TRUE = FALSE
FALSE $\land$ FALSE = FALSE

Truth Table for $\land$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x \land y$</th>
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<tbody>
<tr>
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TRUE ∨ TRUE = TRUE
TRUE ∨ FALSE = TRUE
FALSE ∨ TRUE = TRUE
FALSE ∨ FALSE = FALSE

Truth Table for ∨

<table>
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<tr>
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</table>
\[ \neg \text{TRUE} = \text{FALSE} \]
\[ \neg \text{FALSE} = \text{TRUE} \]

**Truth Table for \( \neg \)**

<table>
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<tr>
<th>( x )</th>
<th>( \neg x )</th>
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<tbody>
<tr>
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</table>
Example
Calculate the value of

\[
\text{TRUE} \land ((\neg \text{FALSE}) \lor \text{FALSE})
\]

Proposition 7.1: DeMorgan’s Law
The Boolean expressions \(\neg (x \land y)\) and \((\neg x) \lor (\neg y)\) are logically equivalent.

Proof Template 4: Truth Table of Logical Equivalence
To show that two Boolean expressions are logically equivalent:

- Construct a truth table showing the values of the two expressions for all possible values of the variables.
- Check to see that the two Boolean expressions always have the same value.
Theorem 7.2: Properties of Boolean Expressions

- $x \land y = y \land x$ and $x \lor y = y \lor x$
- $(x \land y) \land z = x \land (y \land z)$ and $(x \lor y) \lor z = x \lor (y \lor z)$
- $x \land \text{TRUE} = x$ and $x \lor \text{FALSE} = x$
- $\neg(\neg x) = x$
- $x \land x = x$ and $x \lor x = x$
- $x \land (y \lor z) = (x \land y) \lor (x \land z)$ and $x \lor (y \land z) = (x \lor y) \land (x \lor z)$
- $x \land (\neg x) = \text{FALSE}$ and $x \lor (\neg x) = \text{TRUE}$
- $\neg(x \land y) = (\neg x) \lor (\neg y)$ and $\neg(x \lor y) = (\neg x) \land (\neg y)$
### Truth Table for →

<table>
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### Truth Table for ↔

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Homework

- **To Turn In:** p. 24 (1–4, 6, 9b, 11) and p. 28 (1, 3, 11b, 13b)

- **To Discuss:** p. 24 (1, 3, 6, 9b) and p. 28 (1bc, 3, 11b)