Math 5301 Homework 2

1. Consider the measurable space $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$. Prove that the following sets are elements of $\mathcal{B}$:
   (a) $[a, b)$, for any real numbers $a < b$
   (b) $[a, \infty)$, for any $a \in \mathbb{R}$
   (c) $\mathbb{N}$
   (d) $\mathbb{Q}$
   (e) The set of all irrational numbers in $[0, 1]$.
2. Let μ be Lebesgue measure on $(\mathbb{R}, \mathcal{B})$, and find the Lebesgue measure of each of the sets listed in Problem 1.
3. Consider the measurable space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, and let μ be a measure on this space, such that $\mu(\{x\}) = 2^{-x}$, for all $x \in \mathbb{N}$.
   (a) Prove that $\mu(\mathbb{N}) = 1$.
   (b) Calculate $\mu(\{1, 2, 3\})$.
   (c) Calculate $\mu(\{3, 4, 5, \ldots\})$.
4. Let $f : U \to V$, and let $A_i \subseteq V$, for $i = 1, 2, 3, \ldots$. Prove the following:
   (a)
   $$f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(A_i)$$
   (b) If $A_1, A_2, \ldots$ are pairwise disjoint, then $f^{-1}(A_1), f^{-1}(A_2), \ldots$ are pairwise disjoint.
5. Let $(\Omega_1, \mathcal{F}_1, \mu)$ be a measure space and $(\Omega_2, \mathcal{F}_2)$ be a measurable space, and suppose $f : \Omega_1 \to \Omega_2$ is a measurable function. Prove that the following function is a measure on $(\Omega_2, \mathcal{F}_2)$:

$$\mu_f : \mathcal{F}_2 \to [0, \infty]$$
$$\mu_f(A) = \mu(f^{-1}(A)).$$