Math 5301 Homework 3

1. Suppose $(S, \mathcal{F}, P)$ is a probability space and $A, B \in \mathcal{F}$. Prove that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

   (This is called the *inclusion exclusion formula*.)

2. Suppose $(S, \mathcal{P}(S), P)$ is a probability space where

   - S is a finite set, and
   - there exists a constant $c \in [0, 1]$, such that $P(\{s\}) = c$, for all $s \in S$ (the outcomes in S are equally likely).

   (a) Prove that $c = \frac{1}{|S|}$, where $|S|$ denotes the number of elements in S.

   (b) Prove that $P(A) = \frac{|A|}{|S|}$, for any $A \subseteq S$.

   This distribution is called the *discrete uniform* distribution on S.

3. Consider the discrete uniform distribution P on the set

$$S = \{(s_1, s_2) \mid s_1, s_2 \in \{1, 2, 3, 4, 5, 6\}\}.$$

   This is a model for rolling two fair, independent, six-sided dice. The sum of the die rolls is the random variable

$$X : S \to \mathbb{R}$$
$$X(s_1, s_2) = s_1 + s_2.$$

   Find the support and probability mass function of X. Is X a discrete random variable?

4. Verify that the m.g.f. of a Poisson distribution with parameter λ is

$$M(t) = e^{\lambda(e^t - 1)}, \text{ for } -\infty < t < \infty.$$

   Use this fact to prove that $\mu = \sigma^2 = \lambda$ for the Poisson distribution. (Hint: Recall that $e^c = \sum_{x=0}^{\infty} \frac{c^x}{x!}$, for any $c \in \mathbb{R}$.)

5. Verify that the m.g.f. of an exponential distribution with parameter θ is

$$M(t) = \frac{1}{1 - \theta t}, \text{ for } t < \frac{1}{\theta}.$$

   Use this fact to prove that $\mu = \theta$ and $\sigma^2 = \theta^2$ for the exponential distribution.