Math 5301 Homework 5

1. The gas mileage values for a certain type of car are normally distributed with mean μ, and in a sample of 20 of these cars, $\overline{X} = 32.4$ mpg, and $S = 4.8$ mpg.

   (a) Find a 95% confidence interval for μ.

   (b) Perform the hypothesis test $\mathrm{H}_0 : \mu = 35$ vs. $\mathrm{H}_1 : \mu \neq 35$ at the $\alpha = 0.05$ significance level.

   (c) What is the value of the test statistic?

   (d) What is the critical region for this test?

   (e) Find the p-value for this test.

2. Let $X_1, \ldots, X_n$, $n \geq 30$, be a random sample from a probability distribution (not necessarily normal) with finite mean μ and finite positive variance σ^2. For the testing problem

$$\mathrm{H}_0 : \mu = \mu_0 \text{ vs. } \mathrm{H}_1 : \mu \neq \mu_0,$$

   consider the test statistic

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}},$$

   with decision rule

$$\text{Reject } \mathrm{H}_0, \text{ if } |Z| \geq z_{\alpha/2}.$$

   (a) Prove that the null distribution of Z is approximately $N(0, 1)$.

   (b) What is the critical region for this test?

   (c) Prove that the significance level of this test is approximately α.

   (d) Prove that an approximate $1 - \alpha$ confidence interval for μ is given by

$$\left[\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right].$$

3. A random variable X has a bernoulli distribution with parameter $p \in [0, 1]$ if

$$P(X = 1) = p, \text{ and } P(X = 0) = 1 - p.$$

   For a random sample $X_1, \ldots, X_n \sim$ bernoulli(p),

$$\sum_{i=1}^{n} X_i \sim \text{binomial}(n, p).$$

   The maximum likelihood estimator for p is

$$\hat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

   For the testing problem $\mathrm{H}_0 : p = p_0$ vs. $\mathrm{H}_1 : p \neq p_0$, consider the test statistic

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}},$$

   with decision rule

$$\text{Reject } \mathrm{H}_0, \text{ if } |Z| \geq z_{\alpha/2}.$$

(a) Prove that the null distribution of Z is approximately $N(0, 1)$.

(b) What is the critical region for this test?

(c) Prove that the significance level of this test is approximately α.

4. In a random sample of 2000 voters, 827 approved of a presidential candidate. Let $p \in [0, 1]$ be the proportion of the population who approve of this candidate (the approval rating).

   (a) Test $\mathrm{H}_0 : p = 0.4$ vs. $\mathrm{H}_1 : p \neq 0.4$ at the $\alpha = 0.05$ significance level.

   (b) What is the p-value?