Math 5364 Homework 11

Do p. 105 (1, 9, 15) from *Advanced Calculus* by Folland.

1. Let $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^m$, and $y \in \mathbb{R}^n$. Show that

$$\frac{\partial}{\partial x} x' A y = A y$$

2. If $\Sigma \in \mathbb{R}^{n \times n}$ is symmetric, show that

$$\frac{\partial}{\partial x} x' \Sigma x = 2 \Sigma x$$

3. Let $X \in \mathbb{R}^{n \times p}$ be a full rank matrix, and let $Y \in \mathbb{R}^n$. Find the vector $\gamma \in \mathbb{R}^p$ that minimizes $\|Y - X\gamma\|^2$. (Note that this is not a Lagrange multipliers problem, because there is no constraint on γ.)

4. Bonus: Let $\Sigma_{11} \in \mathbb{R}^{p_1 \times p_1}$ and $\Sigma_{22} \in \mathbb{R}^{p_2 \times p_2}$ be positive definite matrices. Also, let $\Sigma_{12} \in \mathbb{R}^{p_1 \times p_2}$ and define $\Sigma_{21} = \Sigma_{12}'$. If

$$c_1 = \max\{x' \Sigma_{12} y \mid x' \Sigma_{11} x = 1, y' \Sigma_{22} y = 1\},$$

is attained at (x, y), show that

$$\begin{pmatrix} -c_1 \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & -c_1 \Sigma_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

This is the main result from the theory of canonical correlations. This maximum is

$$c_1 = \max\{\text{cov}(x'U, y'V) \mid \text{Var}(x'U) = 1, \text{Var}(y'V) = 1\},$$

where U and V are random vectors with joint covariance matrix

$$\text{cov}\left[\begin{pmatrix} U \\ V \end{pmatrix}\right] = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$