Math 5364 Notes

Lagrange Multipliers

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Gradient of a function

- Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function.
- The gradient of f is

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)'$$

Lagrange Multipliers Method

- Let f and G be continuously differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$.
- Define the surface $S = \{x \in \mathbb{R}^n \mid G(x) = 0\}$.
- Assume $\nabla G \neq 0$ on S.
- For $\lambda \in \mathbb{R}$, define the *Lagrangian*

$$L(x, \lambda) = f(x) - \lambda G(x)$$

- Consider the maximization (or minimization) problem

$$\max\{f(x) \mid G(x) = 0\}.$$

- If the maximum is attained at some point x_0, then at that point,

$$\frac{\partial}{\partial x_i} L = 0, \text{ for all } i = 1, \ldots, n$$

$$\frac{\partial}{\partial \lambda} L = 0$$

Example

Find the extreme values of $f(x, y) = x^2 + y^2 + y$ on the *circle* $x^2 + y^2 = 1$.

- $f(x, y) = x^2 + y^2 + y$ and $G(x, y) = x^2 + y^2 - 1$

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- So, $x = 0$, and $y = \pm 1$.

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- The minimum value of 0 is attained at $(0, -1)$.

Proposition

- Let $D \subseteq \mathbb{R}^n$.
- Let $f : D \to \mathbb{R}$.
- If x_0 is an interior point of D, f is differentiable at x_0, and f has a global or local max/min at x_0, then

$$\nabla f(x_0) = 0.$$

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Find the extreme values of $f(x, y) = x^2 + y^2 + y$ on the *disk* $x^2 + y^2 \leq 1$.

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- $f(0, 1) = 2$, $f(0, -1) = 0$, and $f(0, -\frac{1}{2}) = -\frac{1}{4}$.

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- The absolute maximum value of 2 is attained at $(0, 1)$.
- The absolute minimum value of $-\frac{1}{4}$ is attained at $(0, -\frac{1}{2})$.

Partial Derivative Vector Notation

- Let $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$.
- Consider a function $f(x, y)$, that is, $f : \mathbb{R}^{m+n} \to \mathbb{R}$.
- Define

$$\frac{\partial}{\partial x} f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m} \right)'$$

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Examples

- Let $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^m$, and $y \in \mathbb{R}^n$.

$$\frac{\partial}{\partial x} x' A y = A y$$

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- If $\Sigma \in \mathbb{R}^{n\times n}$ is symmetric, then

$$\frac{\partial}{\partial x} x' \Sigma x = 2\Sigma x$$

- Special cases: If $x, y \in \mathbb{R}^n$, then

$$\frac{\partial}{\partial x} x' y = y$$

$$\frac{\partial}{\partial y} x' y = x$$

$$\frac{\partial}{\partial x} x' x = 2x$$

Example

- Let $\Sigma \in \mathbb{R}^{n \times n}$ be a symmetric matrix.
- Show that

$$\max\{x'\Sigma x \mid x'x = 1\}$$

is attained at an eigenvector of Σ. Show that the maximum value is the corresponding eigenvalue.

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$$x'\Sigma x = x'\lambda x = \lambda$$

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$$\nabla f(x_0) = \frac{\partial}{\partial x} f(x_0) = 0.$$

- Let f and G_k, $k = 1, \dots, K$ be continuously differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$.
- Define the surface $S = \{x \in \mathbb{R}^n \mid G_1(x) = \cdots = G_K(x) = 0\}$.
- Assume $\nabla G_k \neq 0$ on S for all k.
- For $\lambda_1, \dots, \lambda_K \in \mathbb{R}$, define the *Lagrangian*

$$L(x, \lambda) = f(x) - \lambda_1 G_1(x) - \cdots - \lambda_K G_K(x)$$

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