Math 505
Some Notes on Continuous Random Variables

Definition 0.1. A random variable X is said to be *continuous* if there exists a function $f : \mathbb{R} \to [0, \infty)$ such that, for any real numbers $a < b$,

$$P(a < X < b) = \int_a^b f(x)\ dx.$$

The function f is called the *probability density function* (p.d.f.) of X.

Any p.d.f. must satisfy these properties

- $\int_{-\infty}^{\infty} f(x)\ dx = 1$
- $f(x) \geq 0$ for (almost) every $x \in \mathbb{R}$.[1]

Example 0.2. Let $\theta > 0$. Then X has an *exponential distribution* with parameter θ if its p.d.f. is

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0. \end{cases}$$

For instance, if X has an exponential distribution with $\theta = 5$, then

$$P(2 < X < 7) = \int_2^7 \frac{1}{5} e^{-x/5}\ dx = e^{-2/5} - e^{-7/5} = 0.424.$$

Definition 0.3. Suppose X is a continuous random variable with p.d.f. f. Then X is *integrable* if

$$\int_{-\infty}^{\infty} |x| f(x)\ dx \text{ is finite.}$$

If X is integrable, then its *expected value* is

$$E(X) = \int_{-\infty}^{\infty} x f(x)\ dx.$$

If X is *not integrable*, then its expected value does not exist.

Example 0.4. If X has an exponential distribution with parameter θ, then

$$\begin{aligned} \int_{-\infty}^{\infty} |x| f(x)\ dx &= \int_{-\infty}^{0} |x| f(x)\ dx + \int_0^{\infty} |x| f(x)\ dx \\ &= \int_{-\infty}^{0} |x| \cdot 0\ dx + \int_0^{\infty} x \cdot \tfrac{1}{\theta} e^{-x/\theta}\ dx \\ &= \theta. \end{aligned}$$

In the second integral, $0 < x < \infty$, so $|x| = x$, and we integrate by parts to finish the calculation. This shows that $\int_{-\infty}^{\infty} |x| f(x)\ dx$ is finite, so X is integrable. Its expected value is

[1] f could be negative on a set of "measure zero", such as a finite set or a countably infinite set, but there is always a version of f that is nonnegative for every $x \in \mathbb{R}$, so this technicality is usually not important.

$$E(X) = \int_{-\infty}^{\infty} x f(x)\ dx = \theta,$$

using the same approach.

Example 0.5. Let X have a *Cauchy distribution*, given by the p.d.f.

$$f(x) = \frac{1}{\pi(1+x^2)}, \text{ for } x \in \mathbb{R}.$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty} |x| f(x)\ dx &= \int_{-\infty}^{\infty} \frac{|x|}{\pi(1+x^2)}\ dx \\ &= 2\int_{0}^{\infty} \frac{x}{\pi(1+x^2)}\ dx = \infty.\end{aligned}$$

Here, we are using symmetry of the function to rewrite the integral, which can be calculated with the substitution $u = 1+x^2$. Since this integral is infinite, a random variable with a Cauchy distribution is not integrable. Therefore, its expected value does not exist, by definition.

Definition 0.6. Suppose X and Y are two random variables, and $f : \mathbb{R}^2 \to [0, \infty)$, such that for all real numbers $a < b$ and $c < d$,

$$P(a < X < b \text{ and } c < Y < d) = \int_a^b \int_c^d f(x, y)\ dydx.$$

Then f is called the *joint probability density function* of X and Y.

Any joint p.d.f. must satisfy these properties

- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\ dydx = 1$
- $f(x, y) \geq 0$ for (almost) every pair $(x, y) \in \mathbb{R}^2$.

Example 0.7. Let X and Y have joint p.d.f.

$$f(x, y) = \begin{cases} \frac{1}{2}e^{-2x} & \text{if } 0 < x < \infty, -e^x < y < e^x \\ 0 & \text{otherwise.} \end{cases}$$

This function is nonnegative, so to verify that it's a p.d.f., we perform the following calculation

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\ dydx &= \int_0^{\infty}\int_{-e^x}^{e^x} \tfrac{1}{2}e^{-2x}\ dydx \\ &= \int_0^{\infty} [\tfrac{1}{2}e^{-2x}y]_{-e^x}^{e^x}\ dx \\ &= \int_0^{\infty} \tfrac{1}{2}[e^{-2x}e^x - (-e^{-2x}e^x)]\ dx \\ &= \int_0^{\infty} e^{-x}\ dx = 1.\end{aligned}$$

Proposition 0.8. If X and Y have a joint p.d.f. f, then the (marginal) p.d.f. of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\ dy.$$

Example 0.9. Continuing the above example, the marginal p.d.f. of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\ dy.$$

If $x \leq 0$, then $f(x,y) = 0$, the above integral will be zero, and $f_X(x) = 0$. If $x > 0$, then this integral is

$$f_X(x) = \int_{-e^x}^{e^x} \tfrac{1}{2}e^{-2x}\ dy = e^{-x}.$$

Therefore, the marginal p.d.f. of X is

$$f_X(x) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0, \end{cases}$$

so X has an exponential distribution with parameter $\theta = 1$.

Definition 0.10. Suppose X and Y have joint p.d.f. f and X has marginal p.d.f. f_X. Then the conditional p.d.f. of Y given $X = x$ is

$$f(y \mid x) = \frac{f(x,y)}{f_X(x)}.$$

The conditional expected value of Y given $X = x$ is then

$$E(Y \mid X = x) = \int_{-\infty}^{\infty} y f(y \mid x)\ dy,$$

assuming that the integral *converges absolutely*, meaning

$$\int_{-\infty}^{\infty} |y| f(y \mid x)\ dy \text{ is finite.}$$

Example 0.11. For the above example,

$$f(y \mid x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{1}{2}e^{-2x}}{e^{-x}} = \tfrac{1}{2}e^{-x}, \text{ for } 0 < x < \infty, -e^x < y < e^x \tag{0.1}$$

$$E(Y \mid X = x) = \int_{-\infty}^{\infty} y f(y \mid x)\ dy = \int_{-e^x}^{e^x} y \cdot \tfrac{1}{2}e^{-x}\ dy = 0, \text{ for } 0 < x < \infty.$$

It's easy to check that this integral converges absolutely, so the conditional expected value does indeed exist. Let's take a moment to consider what this means.

The p.d.f. of X is only positive when $x > 0$, so with probability 1, X is positive. Assuming X has taken a particular value x, which is positive, the p.d.f. for Y is given in equation (0.1). Now, this p.d.f. does not depend at all on y. In other words, the conditional distribution of Y given $X = x$ is uniform on the interval $(-e^x, e^x)$. Since an interval of this form is symmetric about 0, it makes sense that the conditional expected value of Y given $X = x$ is 0, regardless of what x is.

What about $E(Y)$, the unconditional expected value of Y? This only exists if Y is integrable, meaning that the following integral is finite.

$$\begin{aligned}
\int_{-\infty}^{\infty} |y| f_Y(y)\ dy &= \int_{-\infty}^{\infty} |y| \left[\int_{-\infty}^{\infty} f(x,y)\ dx \right] dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |y| f(x,y)\ dydx \\
&= \int_{0}^{\infty} \int_{-e^x}^{e^x} |y| \cdot \tfrac{1}{2} e^{-2x}\ dydx \\
&= \int_{0}^{\infty} \tfrac{1}{2} e^{-2x} \int_{-e^x}^{e^x} |y|\ dydx \\
&= \int_{0}^{\infty} \tfrac{1}{2} e^{-2x} \cdot 2 \int_{0}^{e^x} y\ dydx \\
&= \int_{0}^{\infty} \tfrac{1}{2} e^{-2x} e^{2x}\ dx \\
&= \int_{0}^{\infty} \tfrac{1}{2}\ dx = \infty.
\end{aligned}$$

Therefore, we have just seen an example where $E(Y \mid X) = 0$ for all possible values of X, but Y is not integrable, so $E(Y)$ does not exist.