Calculus II Review Four

For problems 1 through 9, determine whether or not the series converges, and state which convergence test was used.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{5^n}$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n+6}$

3. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4(n!)}$

6. $\displaystyle\sum_{n=7}^{\infty} \frac{\sqrt{1+n^3}}{n-6}$

7. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n^3+1)^n}{n^{2n}}$

10. Find the interval and radius of convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}(x-5)^n$.

Answers

1. This is a geometric series with $r = \frac{1}{5}$ (see p. 706). Since $|r| < 1$, the series converges to
$$\frac{1}{1-\frac{1}{5}} = \frac{5}{4}.$$
(Note that $\sum_{n=1}^{\infty} r^{n-1} = \sum_{n=0}^{\infty} r^n$, so the expressions for geometric series given in the book and in class are equivalent).

2. Note that $\frac{n}{n+6} \to 1$. Because $\frac{n}{n+6} \not\to 0$, the series $\sum_{n=0}^{\infty} \frac{n}{n+6}$ is divergent by the divergence test (p. 709).

3. The function $f(x) = \frac{1}{x(\ln x)^2}$ is a continuous, positive, decreasing function on the interval $[2, \infty)$. By the integral test (p. 716), the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges if and only if the integral $\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx$ converges.

   Using the substitution $u = \ln x$, which yields $du = \frac{1}{x}\, dx$, the integral is
$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx = \int_{\ln 2}^{\infty} \frac{1}{u^2}\, du = -\frac{1}{u}\bigg|_{\ln 2}^{\infty} = \frac{1}{\ln 2}.$$
   Since the integral converges, the series converges by the integral test.

4. This is a p-series with $p = 1$ (p. 717). Since $p \leq 1$, the series diverges.

5. On this problem, we use the comparison test (p. 722). Note that $\frac{1}{n^4(n!)}$ and $\frac{1}{n^4}$ both have positive terms, and
$$\frac{1}{n^4(n!)} \leq \frac{1}{n^4}, \text{ for all } n \geq 1.$$
   Also, $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a p-series with $p = 4$, so it converges. Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^4(n!)}$ converges by the comparison test.

6. Notice that
$$\frac{\sqrt{1+n^3}}{n-6} \approx \frac{\sqrt{n^3}}{n} = \frac{n^{3/2}}{n} = \frac{1}{n^{-1/2}}.$$
   This suggests comparing $\sum_{n=7}^{\infty} \frac{\sqrt{1+n^3}}{n-6}$ to $\sum_{n=7}^{\infty} \frac{1}{n^{-1/2}}$ using the limit comparison test (p. 724).
$$\frac{\frac{\sqrt{1+n^3}}{n-6}}{\frac{1}{n^{-1/2}}} \to 1.$$
   By the limit comparison test, $\sum_{n=7}^{\infty} \frac{\sqrt{1+n^3}}{n-6}$ and $\sum_{n=7}^{\infty} \frac{1}{n^{-1/2}}$ either both converge or both diverge. Since $\sum_{n=7}^{\infty} \frac{1}{n^{-1/2}}$ diverges (it's a p-series with $p = -\frac{1}{2}$), the original series diverges.

7. This is an alternating series (p. 727) with $b_n = \frac{1}{n}$, and

   - $$b_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = b_n, \text{ for all } n$$
   - $$b_n = \frac{1}{n} \to 0.$$

   Therefore, the series converges by the alternating series test.

8. The factors 3^n and $n!$ suggest using the ratio test (p. 734).

   $$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{3^{n+1}(n+1)^2}{(n+1)!}}{\frac{3^n n^2}{n!}}\right| = \left|\frac{3^{n+1}(n+1)^2 n!}{3^n n^2 (n+1)!}\right| = \frac{3(n+1)^2}{(n+1)n^2} \to 0,$$

   since the degree of the denominator is greater than the degree of the numerator. Because the limit is less than 1, the series converges by the ratio test.

9. The fact that both $2n^3 + 1$ and n have been raised to the power of n suggests using the root test (p. 736).

   $$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{(2n^3+1)^n}{n^{2n}}} = \frac{2n^3+1}{n^2} \to \infty,$$

   since the degree of the numerator is greater than the degree of the denominator. Because the limit is greater than 1, the series diverges by the root test.

10. To find the interval of convergence of a power series, we always start with the ratio test.

    $$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{n+1}(x-5)^{n+1}}{\frac{1}{n}(x-5)^n}\right| = \frac{n}{n+1}|x-5| \to |x-5|.$$

    According to the ratio test, the series converges if $|x - 5| < 1$, so the radius of convergence is 1 and the series converges on the interval $(4, 6)$. We need to check the endpoints 4 and 6 separately.

    Plugging in $x = 4$ yields the series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$, which converges by the alternating series test.

    Plugging in $x = 6$ yields the series $\sum_{n=1}^{\infty} \frac{1}{n}$, which is a divergent p-series.

    Therefore, the interval of convergence is $[4, 6)$ and the radius is 1.