Math 505 Notes

Chapter 1

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Outline

Definition

Let $(S, \mathcal{B}, P)$ be a probability space. A *random variable* on S is a function

$$X : S \to \mathbb{R}$$

- such that $X^{-1}(B) \in \mathcal{B}$ for every $B \in \mathcal{B}_0$.
- ($\mathcal{B}_0$ is the Borel σ-field on $\mathbb{R}$.)

Notation

Given $B \in \mathcal{B}_0$,

- $[X \in B] := \{s \in S \,|\, X(s) \in B\}$
- $P[X \in B] = P(\{s \in S \,|\, X(s) \in B\})$

X induces a probability measure P_X on $(\mathbb{R}, \mathcal{B}_0)$ given by

- $P_X(B) = P[X \in B]$
- P_X is the *distribution* of X

Definition

- The space of X is the set of all possible values of X,
- $\mathcal{D} = \{X(s) \mid s \in S\}$

Example

- Roll two fair dice independently
- $S = \{(s_1, s_2) \mid s_1, s_2 \in \{1, \ldots, 6\}\}$
- X = sum of die rolls
  - $X : S \to \mathbb{R}$
  - $X(s_1, s_2) = s_1 + s_2$
- X is a random variable with space $\mathcal{D} = \{2, 3, \ldots, 12\}$

Definition

The *cumulative distribution function* of X is

- $F : \mathbb{R} \to [0, 1]$
- $F(x) = P[X \le x]$

Defining properties of a c.d.f.

- If $a < b$, then $F(a) \le F(b)$ (F is nondecreasing)
- $\lim_{x\to-\infty} F(x) = 0$
- $\lim_{x\to\infty} F(x) = 1$
- For any $x_0 \in \mathbb{R}$, $\lim_{x\to x_0^+} F(x) = F(x_0)$ (F is right continuous)

- $F(x_0-) := \lim_{x\to x_0^-} F(x)$
- $P(X = x) = F(x) - F(x-)$
- $P(a < X \le b) = F(b) - F(a)$
- $P(a < X < b) = F(b-) - F(a)$

Definition

- If the space of X is countable, X is called a *discrete* random variable,
- and it's probability mass function is
  - $p : \mathbb{R} \to [0, 1]$
  - $p(x) = P[X = x]$

Defining properties of a p.m.f.

- $p(x) \geq 0$, for every $x \in \mathbb{R}$
- $\sum_{x \in \mathbb{R}} p(x) = 1$

Calculating probabilities with a p.m.f.

For a discrete random variable with p.m.f. p,

$$P[X \in B] = \sum_{x \in B} p(x),$$

for every $B \in \mathcal{B}_0$.

Definition

- If the c.d.f. of X is continuous, X is called a *continuous* random variable.
- Implies $P(X = x) = 0$ for every $x \in \mathbb{R}$
- Space of X is typically an interval

Definition

Suppose $f : \mathbb{R} \to [0, \infty)$ such that

$$P[X \in B] = \int_B f(x)dx,$$

for every $B \in \mathcal{B}_0$. Then f is called a *probability density function* for X.

Defining properties of a p.d.f.

- $f(x) \geq 0$, for every $x \in \mathbb{R}$
- $\int_{-\infty}^{\infty} f(x)dx = 1$

- If X has p.m.f. p, then $F(x) = \sum_{t \leq x} p(t)$
- If X has p.d.f. f, then $F(x) = \int_{-\infty}^{x} f(t)dt$
- Implies $F'(x) = f(x)$ on intervals where f is continuous

Definition

If X and Y are two random variables with the same distribution, we write $X \overset{D}{=} Y$.

- $X \overset{D}{=} Y$ iff $F_X = F_Y$
- If X and Y are discrete, $X \overset{D}{=} Y$ iff $p_X = p_Y$
- If X and Y have densities, $X \overset{D}{=} Y$ iff $f_X = f_Y$ (almost everywhere)

Outline

Definition

- Let X be a random variable with p.m.f. p such that
- $\sum_{x \in \mathbb{R}} |x| p(x) < \infty$.
- Then the *expected value* of X is $E(X) = \sum_{x \in \mathbb{R}} x p(x)$.

Definition

- Let X be a random variable with p.d.f. f such that
- $\int_{-\infty}^{\infty} |x| f(x) dx < \infty$.
- Then the *expected value* of X is $E(X) = \int_{-\infty}^{\infty} x f(x) dx$.

Properties of expectation

Let $g : \mathbb{R} \to \mathbb{R}$.

- If X has p.m.f. p, $E[g(X)] = \sum_{x \in \mathbb{R}} g(x)p(x)$ (assuming the sum is absolutely convergent).
- If X has p.d.f. f, $E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$ (assuming the function is integrable).
- If $X \equiv c$, for some $c \in \mathbb{R}$, then $E(X) = c$.
- If $a, b \in \mathbb{R}$, and X and Y are random variables, $E(aX + bY) = aE(X) + bE(Y)$.

Outline

Notation

The expected value of X is often called the *mean*, and denoted $\mu = E(X)$.

Definition

- The *variance* of a random variable X is $\text{Var}(X) = E[(X - \mu)^2]$.
- The *standard deviation* of X is $\sigma = \sqrt{\text{Var}(X)}$.
- $\text{Var}(X) = E(X^2) - E(X)^2$

Definition

Let X be a random variable such that, for some $h > 0$, $E(e^{tX})$ exists for every $-h < t < h$. Then the function

$$M(t) = E(e^{tX})$$

is called the *moment generating function* of X.

Let X be a random variable with m.g.f. M.

- $M(0) = 1$
- $M'(0) = E(X)$
- $M''(0) = E(X^2)$
- $M^{(r)}(0) = E(X^r)$, for any $r = 0, 1, 2, \ldots$
- $\text{Var}(X) = M''(0) - [M'(0)]^2$

Theorem

If X and Y are random variables, $X \stackrel{D}{=} Y$ iff $M_X = M_Y$.

Outline

Theorem

Let X be a random variable and let m and k be integers such that $m \geq k \geq 0$. If $E(X^m)$ exists, then $E(X^k)$ exists.

Markov's Inequality

Let $u : \mathbb{R} \to [0, \infty)$. If $E[u(X)]$ exists, then for every $c > 0$,

$$P[u(X) \geq c] \leq \frac{E[u(X)]}{c}.$$

Chebyshev's Inequality

Suppose X has finite variance σ^2. Then, for every $k > 0$,

$$P[|X - \mu| \geq k\sigma] \leq \frac{1}{k^2}.$$

Definition

A function ϕ defined on an interval (a, b) is said to be *convex* if, for every $x, y \in (a, b)$, and for every $0 < \gamma < 1$,

$$\phi[\gamma x + (1 - \gamma)y] \leq \gamma\phi(x) + (1 - \gamma)\phi(y).$$

- If ϕ is differentiable on (a, b), ϕ is convex iff ϕ is nondecreasing on (a, b).
- If ϕ is twice differentiable on (a, b), ϕ is convex iff $\phi'' \geq 0$ on (a, b).

Jensen's Inequality

If ϕ is convex on an open interval I, and X is a random variable with finite expectation whose support is contained in I, then

$$\phi[E(X)] \leq E[\phi(X)].$$