Math 505 Notes

Chapter 3

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Outline

Definition

Let $p \in [0, 1]$. A *Bernoulli* random variable with parameter p is a random variable X such that

- $P(X = 1) = p$, and
- $P(X = 0) = 1 - p$.

Definition

- Let $n \in \mathbb{N}_+$ and $p \in [0, 1]$,
- and let $X_1, \ldots, X_n$ be independent Bernoulli random variables with parameter p.
- The random variable

  $$X = \sum_{i=1}^{n} X_i$$

  is a *binomial* random variable with parameters n and p,
- denoted $X \sim b(n, p)$.

Characteristics of a Binomial Random Variable

- Let $X \sim b(n, p)$.
- The p.m.f. of X is

$$P[X = x] = \binom{n}{x} p^x (1-p)^{n-x}, \text{ for } x = 0, 1, \ldots, n.$$

- The m.g.f. is

$$M(t) = (1 - p + pe^t)^n, \text{ for } t \in \mathbb{R}.$$

- $E(X) = np$
- $\text{Var}(X) = np(1-p)$

Theorem

- *Let* $X_1, \ldots, X_m$ *be independent random variables such that*
- $X_i \sim b(n_i, p)$, *for* $i = 1, \ldots, m$.
- *Then*

$$\sum_{i=1}^{m} X_i \sim b\left(\sum_{i=1}^{m} n_i, p\right).$$

Theorem

Suppose $X_1, \ldots, X_m$ *are independent random variables with m.g.f.'s* $M_1, \ldots, M_m$. *Then the moment generating function of* $\sum_{i=1}^{m} X_i$ *is given by*

$$M(t) = \prod_{i=1}^{m} M_i(t).$$

Definition

- Consider a sequence of independent Bernoulli trials with $P(\text{Success}) = p$, and let $r \in \mathbb{N}_+$.
- Let Y be the number of failures that occur before the rth success.
- The p.m.f. for Y is

$$P[Y = y] = \binom{y + r - 1}{r - 1} p^r (1 - p)^y, \text{ for } y = 0, 1, \ldots$$

  and Y is said to have a *negative binomial* distribution.
- A negative binomial distribution with $r = 1$ is called a *geometric distribution*.

Definition

- Consider a set of N objects consisting of N_1 red objects and $N - N_1$ blue objects.
- Select n of these objects at random, without replacement, and let X be the number of red objects in the sample.
- Then the p.m.f. of X is

$$P[X = x] = \frac{\binom{N_1}{x}\binom{N-N_1}{n-x}}{\binom{N}{n}},$$

and X is said to have a *hypergeometric distribution*.

Outline

Poisson Process

- X has a *Poisson distribution* with parameter m if

$$P[X = x] = \frac{m^x e^{-m}}{x!}, \text{ for } x = 0, 1, \ldots$$

- Stream of "phone calls"
- Let $g(x, w)$ be the probability of receiving x phone calls in a time interval of length w
- Assumptions:
  - $g(1, h) \approx \lambda h$, for small h
  - $\sum_{x=2}^{\infty} g(x, h) \approx 0$, for small h
  - The number of phone calls in nonoverlapping intervals are independent.
- $$g(x, w) = \frac{(\lambda w)^x e^{-\lambda w}}{x!}$$
- The number of phone calls received in an interval of length w is a Poisson random variable with parameter $m = \lambda w$.

If X has a Poisson distribution with parameter m,

- $M(t) = e^{m(e^t - 1)}$
- $E(X) = m$
- $\text{Var}(X) = m$
- For a Poisson process, the parameter λ represents the average number of “phone calls” in an interval of length 1.
- The average number of “phone calls” in an interval of length w is λw

Outline

A random variable X has an *exponential distribution* with mean $\beta > 0$ if its p.d.f. is

$$f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}, \text{ for } x > 0.$$

The waiting time between "phone calls" in a Poisson process with parameter λ is exponentially distributed with mean $\beta = \frac{1}{\lambda}$.

- Let $\alpha, \beta > 0$.
- $$\Gamma(\alpha) := \int_0^\infty y^{\alpha-1} e^{-y} dy$$
- $\Gamma(\alpha) = (\alpha - 1)!$, for $\alpha \in \mathbb{N}_+$
- If the p.d.f. of X is

  $$\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} \text{ for } x > 0,$$

  then X is said to have a *gamma distribution* with parameters α and *beta*, i.e., $X \sim \Gamma(\alpha, \beta)$.

- Let $\alpha \in \mathbb{N}_+$. The waiting time until the αth "phone call" in a Poisson process with parameter λ has the distribution $\Gamma(\alpha, \frac{1}{\lambda})$.
- If $\alpha \in \mathbb{N}_+$, and $X_1, \ldots, X_\alpha$ are i.i.d. exponential random variables with parameter β, then

$$X_1 + \cdots + X_\alpha \sim \Gamma(\alpha, \beta).$$

- $M(t) = (1 - \beta t)^{-\alpha}$, for $t < \beta^{-1}$.
- $E(X) = \alpha\beta$
- $\mathrm{Var}(X) = \alpha\beta^2$

Definition

- Let $\alpha = \frac{r}{2}$, where $r \in \mathbb{N}_+$, and $\beta = 2$.
- The corresponding gamma distribution is called a χ^2 distribution with r *degrees of freedom*, denoted $\chi^2(r)$.

Theorem

Let $X_1, \ldots, X_n$ be independent random variables.

- *If $X_i \sim \Gamma(\alpha_i, \beta)$, for $i = 1, \ldots, n$, then $\sum_{i=1}^n X_i \sim \Gamma(\sum_{i=1}^n \alpha_i, \beta)$.*
- *If $X_i \sim \chi^2(r_i)$, for $i = 1, \ldots, n$, then $\sum_{i=1}^n X_i \sim \chi^2(\sum_{i=1}^n r_i)$.*

Definition

Let $\alpha, \beta > 0$, and suppose the p.d.f. of X is

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \text{ for } 0 < x < 1.$$

Then X is said to have a *beta* distribution with parameters α and β.

Outline

Definition

A random variable Z with pdf

$$f(z) = \exp\left\{-\frac{z^2}{2}\right\}, \ z \in \mathbb{R}$$

has a *standard normal distribution*.

- Its mean is 0.
- Its variance is 1.

Definition

A random variable X has a normal distribution with mean μ and variance σ^2 if

$$X = \sigma Z + \mu,$$

where Z is a standard normal random variable, i.e., if

$$Z = \frac{X - \mu}{\sigma}$$

has a standard normal distribution.

- $X \sim N(\mu, \sigma^2)$
- Its density is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \ x \in \mathbb{R}.$$

Theorem

If $X \sim N(\mu, \sigma^2)$, *then*

$$V = \frac{(X - \mu)^2}{\sigma^2}$$

has a $\chi^2(1)$ *distribution.*

Theorem

Let $X_1, \ldots, X_n$ *be independent random variables such that* $X_i \sim N(\mu_i, \sigma_i^2)$, *for* $i = 1, \ldots, n$. *Then, given constants* $a_1, \ldots, a_n$,

$$Y = \sum_{i=1}^{n} a_i X_i \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right).$$

Corollary

Let $X_1, \ldots, X_n$ be i.i.d. normally distributed random variables with mean μ and variance σ^2. Then

$$\overline{X} = n^{-1} \sum_{i=1}^{n} X_i \sim N(\mu, \sigma^2/n).$$

Outline

Definition

Let $Z_1, \ldots, Z_n$ be i.i.d. N(0,1) random variables. Then the random vector $Z = (Z_1, \ldots, Z_n)'$ has a multivariate normal distribution with

- mean vector $E(Z) = 0$, and
- covariance matrix $\text{cov}(Z) = I_n$.
- $Z \sim N_n(0, I_n)$
- $$M_Z(t) = \exp\left\{\frac{1}{2}t't\right\}.$$
- $$f_Z(z) = \frac{1}{(2\pi)^{n/2}} \exp\left\{-\frac{1}{2}z'z\right\}, \; z \in \mathbb{R}^n$$

Definition

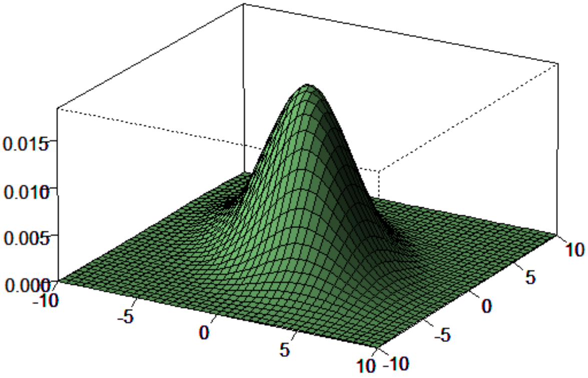
The random vector X has a multivariate normal distribution if

$$X = \Sigma^{\frac{1}{2}} Z + \mu,$$

where $\mu \in \mathbb{R}^n$, Σ is an $n \times n$ positive semi-definite matrix, and $Z \sim N_n(0, I_n)$.

- $E(X) = \mu$
- $\text{cov}(X) = \Sigma$
- $X \sim N_n(\mu, \Sigma)$
- $$M_X(t) = \exp\left\{ t'\mu + \frac{1}{2} t'\Sigma t \right\}$$
- If Σ is positive definite,

$$f_X(x) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp\left\{ -\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu) \right\}, \; x \in \mathbb{R}^n$$


0.015
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-10
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10

Theorem

Suppose $X \sim N_n(\mu, \Sigma)$*,* $A \in \mathbb{R}^{m \times n}$*, and* $b \in \mathbb{R}^m$*. Then*

$$AX + b \sim N_m(A\mu + b, A\Sigma A').$$

Corollary

Suppose $X \sim N_n(\mu, \Sigma)$*, and let* X_1 *and* X_2 *be be random vectors containing the first* n_1 *and last* $n_2 = n - n_1$ *components of* X*, respectively, i.e.,*

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$

Partitioning μ *and* Σ *accordingly, we have*

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \text{ and } \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$

Then $X_1 \sim N_{n_1}(\mu_1, \Sigma_{11})$*, and* $X_2 \sim N_{n_2}(\mu_2, \Sigma_{22})$*.*

Theorem

Let $X = (X_1', X_2')'$*, as in the previous corollary. Then* X_1 *and* X_2 *are independent if and only if* $\Sigma_{12} = 0$*. That is, two jointly normal random vectors (variables) are independent if and only if they are uncorrelated.*

Theorem

Suppose $X = (X_1', X_2')'$*, as in the previous corollary. Then the conditional distribution of* X_1 *given* X_2 *is*

$$N_m(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(X_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}).$$

Outline

Theorem

- $$E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i)$$