Math 5305 Midterm Exam Review

1. Chapters 1 and 2 of Freedman, Pisani, and Purves.
   (a) Know definitions such as randomized controlled double blind experiment, observational study, confounding variable, etc.
   (b) Be prepared to answer questions related to these ideas.

2. Prove the following.
   (a) If $A, B \in \mathbb{R}^{I\times I}$, then trace$(A+B) =$ trace$(A) +$ trace(B).
   (b) If $A \in \mathbb{R}^{I\times J}$, and $B \in \mathbb{R}^{J\times I}$, then trace$(AB) =$ trace(BA).
   (c) If $A \in \mathbb{R}^{I\times J}$, U is a random matrix taking values in $R^{J\times K}$, and $B \in \mathbb{R}^{K\times L}$, then $E(AUB) = AE(U)B$.
   (d) If $A \in \mathbb{R}^{N\times M}$, and U is a random vector of length M, then $\text{Cov}(AU) = A\text{Cov}(U)A'$.

3. Let $X \in \mathbb{R}^{n\times p}$, where $p \leq n$. Show that X has full rank if and only if $X'X$ has full rank.

4. Know the definitions of nonnegative/positive definite and orthogonal matrices, and be prepared to work related problems.

5. Write the equation for a multiple regression model.
   (a) For each vector/matrix appearing in the model, state the following: its dimensions, whether it is observable or unobservable, and whether it is random or constant.
   (b) State all of the mathematical assumptions that apply to this model.
   (c) Define $\hat{\beta}$, $\hat{Y}$, e, and $\hat{\sigma}^2$.
   (d) Prove that $E(\hat{\beta} \mid X) = \beta$.
   (e) Prove that $E(\hat{\sigma}^2 \mid X) = \sigma^2$.
   (f) Prove that $\text{Cov}(\hat{\beta} \mid X) = \sigma^2(X'X)^{-1}$.
   (g) What is $\hat{\text{SE}}(\hat{\beta}_j)$?

6. Given a multiple regression model, define the coefficient of determination, R^2. If the answer to one of the questions below is "no", provide a counterexample.
   (a) If the model is valid, what does an R^2 value near 1 indicate?
   (b) If the model is valid, what does an R^2 value near 0 indicate?
   (c) Does an R^2 value near 1 prove that the model is valid?
   (d) Does an R^2 value near 1 prove a causal relationship exists between X and Y?
   (e) Does an R^2 value near 0 prove that the association between X and Y is weak?