Math 5305 Notes

Chapter 3

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Outline

Matrices and Addition

Definition

- Suppose n and m are positive integers.
- The set of $n \times m$ matrices with real entries is denoted by $\mathbb{R}^{n \times m}$.

Definition

- Suppose $A, B \in \mathbb{R}^{n \times m}$.
- Define $A + B \in \mathbb{R}^{n \times m}$ by

$$(A + B)_{ij} = A_{ij} + B_{ij}, \text{ for } i = 1, \ldots, n \text{ and } j = 1, \ldots, m.$$

Multiplication

Definition

- Suppose $A \in \mathbb{R}^{I \times J}$ and $B \in \mathbb{R}^{J \times K}$.
- Define $AB \in \mathbb{R}^{I \times K}$ by

$$(AB)_{ik} = \sum_{j=1}^{J} A_{ij} B_{jk}, \text{ for } i = 1, \ldots, I \text{ and } k = 1, \ldots, K.$$

Proposition

Consider matrices $A \in \mathbb{R}^{I \times J}$, $B \in \mathbb{R}^{J \times K}$, $C \in \mathbb{R}^{K \times L}$, and $D \in \mathbb{R}^{L \times M}$. Then, for any $i = 1, \ldots, I$ and $m = 1, \ldots, M$,

$$(ABCD)_{im} = \sum_{j=1}^{J} \sum_{k=1}^{K} \sum_{\ell=1}^{L} A_{ij} B_{jk} C_{k\ell} D_{\ell m}.$$

Transpose and Trace

Definition

- Suppose $A \in \mathbb{R}^{n \times m}$.
- Define $A' \in \mathbb{R}^{m \times n}$ by

$$(A')_{ji} = A_{ij}, \text{ for } i = 1, \ldots, n \text{ and } j = 1, \ldots, m.$$

- If $A' = A$, then A is called *symmetric*.

Definition

- Suppose $A \in \mathbb{R}^{n \times n}$.
- Define the trace of A, $\text{trace}(A)$ by

$$\text{trace}(A) = \sum_{i=1}^{n} A_{ii}.$$

Inner Products and Norms

Definition

Given two vectors $u, v \in \mathbb{R}^n$, their *inner product* is

$$u \cdot v = u'v = u_1 v_1 + \cdots + u_n v_n.$$

Definition

The *norm*, *length*, or *magnitude* of a vector $u \in \mathbb{R}^n$ is

$$\|u\| = \sqrt{u'u} = \sqrt{u_1^2 + \cdots + u_n^2}.$$

Outline

Determinants

- If A is a square matrix, its *determinant* is denoted by $\det(A)$ or $|A|$.
- Examples:

$$\begin{vmatrix} 1 & 2 \\ 5 & 3 \end{vmatrix} = 1 \cdot 3 - 5 \cdot 2 = -7$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix}$$

$$= 1 \cdot 2 - 2 \cdot 2 + 3 \cdot 2 = 4$$

Inverses and Kernels

Definition

An $n \times n$ matrix A is *invertible* if there exists an $n \times n$ matrix A^{-1}, such that

$$AA^{-1} = A^{-1}A = I.$$

Definition

The *kernel* of an $n \times m$ matrix A is

$$\ker(A) = \{v \in \mathbb{R}^m \mid Av = 0\}.$$

Linear Independence and Rank

Definition

- Suppose $v_1, v_2, \ldots, v_k$ are vectors.
- They are *linearly independent* if, for any scalars $c_1, c_2, \ldots, c_k$,

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0 \text{ implies } c_1 = c_2 = \cdots = c_k = 0.$$

Definition

- The *rank* of a matrix is the maximum number of linearly independent columns it has.
- If X is an $n \times p$ matrix, and $\text{rank}(X) = p$, then X has *full rank*.

Proposition

The rank of a matrix is the number of nonzero rows it has in reduced row echelon form.

The Big Theorem for Square Matrices

Theorem

For an $n \times n$ matrix A, the following are equivalent:

- $\det(A) \neq 0$
- A is invertible
- $\ker(A) = \{0\}$
- For any $c \in \mathbb{R}^n$,

$$Ac = 0 \text{ implies } c = 0$$

- All of the columns of A are linearly independent
- $\text{rank}(A) = n$
- A has full rank

The Big Theorem for Nonsquare Matrices

Theorem

For an $n \times p$ matrix X, the following are equivalent:

- $\ker(X) = \{0\}$
- For any $c \in \mathbb{R}^p$,
$$Xc = 0 \text{ implies } c = 0$$
- All of the columns of X are linearly independent
- $\operatorname{rank}(X) = p$
- X has full rank

Outline

Covariance Between Two Random Variables

Definition

- Let X and Y be two random variables.
- The *covariance* between X and Y is

$$\text{cov}(X, Y) = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y)$$

- The covariance measures the strength of the association between X and Y, and the sign indicates whether the relationship is positive or negative.

Correlation Between Two Random Variables

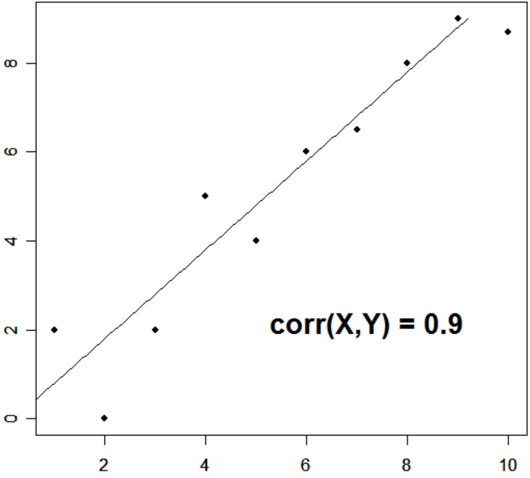
Definition

- The *correlation coefficient* between X and Y is

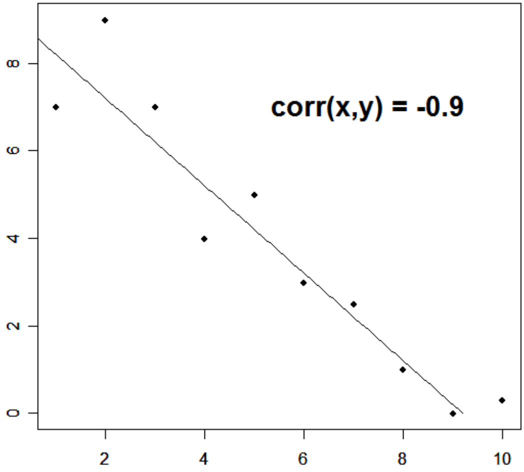
$$\rho = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y}.$$

- $-1 \leq \rho \leq 1$
- Values of ρ near 1 indicate a strong positive relationship.
- Values of ρ near -1 indicate a strong negative relationship.
- Values of ρ near 0 indicate a weak or nonlinear relationship.

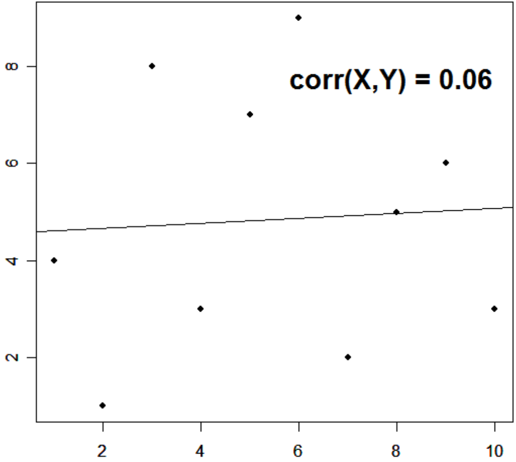
Strong Positive Correlation



Strong Negative Correlation



Virtually No Correlation



Random Vectors

Definition

- A random vector is a vector whose components are random variables.
- If $U_1, \dots, U_n$ are random variables, then

$$U = \begin{pmatrix} U_1 \\ \vdots \\ U_n \end{pmatrix}$$

  is a random vector.

Expected Value of a Random Vector

Definition

- Given a random vector

$$U = \begin{pmatrix} U_1 \\ \vdots \\ U_n \end{pmatrix}$$

  the *expected value* of U is

$$E(U) = \begin{pmatrix} E(U_1) \\ \vdots \\ E(U_n) \end{pmatrix}$$

- $$[E(U)]_i = E(U_i), \text{ for every } i$$

Expected Value of a Random Matrix

Definition

- Given a random matrix

$$U = \begin{pmatrix} U_{11} & \cdots & U_{1m} \\ \vdots & & \vdots \\ U_{n1} & \cdots & U_{nm} \end{pmatrix}$$

  the *expected value* of U is

$$E(U) = \begin{pmatrix} E(U_{11}) & \cdots & E(U_{1m}) \\ \vdots & & \vdots \\ E(U_{n1}) & \cdots & E(U_{nm}) \end{pmatrix}$$

- $$[E(U)]_{ij} = E(U_{ij}), \text{ for every } i, j$$

Covariance Matrix of a Random Vector

Definition

- Given a random vector

$$U = \begin{pmatrix} U_1 \\ \vdots \\ U_n \end{pmatrix}$$

the *covariance matrix* of U is

$$\text{cov}(U) = E\left\{\begin{pmatrix} U_1 - E(U_1) \\ \vdots \\ U_n - E(U_n) \end{pmatrix}(U_1 - E(U_1), \ldots, U_n - E(U_n))\right\}$$

More on Covariance

- $$\text{cov}(U) = E[(U - E(U))(U - E(U))'] = E(UU') - E(U)E(U)'$$
- The ith diagonal element of $\text{cov}(U)$ is $\text{Var}(U_i)$.
- The (i,j) entry of $\text{cov}(U)$ is $\text{cov}(U_i, U_j)$.

Outline

Positive Definite Matrices

Definition

An $n \times n$ matrix G is *non-negative definite* if

- G is symmetric, and
- $x'Gx \geq 0$, for all $x \in \mathbb{R}^n$.

Definition

An $n \times n$ matrix G is *positive definite* if

- G is symmetric, and
- $x'Gx > 0$, for all nonzero $x \in \mathbb{R}^n$.

- Note that any positive definite matrix is non-negative definite.
- We will denote the set of $n \times n$ positive definite matrices by $\mathbf{PD}(n)$.

Diagonal and Orthogonal Matrices

Definition

A matrix D is *diagonal* if all of its entries off the diagonal are zero,

$$D_{ij} = 0 \text{ when } i \neq j.$$

Definition

- A matrix R is *orthogonal* if $R'R = I$.
- If R is orthogonal, $R^{-1} = R'$, and $RR' = I$.

Diagonalizing a Positive Definite Matrix

Theorem

- *G is non-negative definite iff there exists*
- *a diagonal matrix D whose diagonal entries are non-negative and*
- *an orthogonal matrix R,*
- *such that $G = RDR'$*

- This theorem also holds if we replace both instances of "non-negative" with "positive".
- The columns of R are the *eigenvectors* of G.
- The diagonal entries of D are the *eigenvalues* of G.

Eigenvalues and Eigenvectors

Definition

Suppose $A \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^n$ is nonzero, and $\lambda \in \mathbb{R}$, such that

$$Ax = \lambda x.$$

- Then λ is an *eigenvalue* of A, and
- x is an *eigenvector* of A corresponding to λ.

Outline

Definition (Univariate Normal Distribution)

- The *normal distribution* on $\mathbb{R}$ with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$ is given by the p.d.f.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \text{ for } x \in \mathbb{R}.$$

- Denoted by $N(\mu, \sigma^2)$
- If $X \sim N(\mu, \sigma^2)$, then $E(X) = \mu$, and $\text{Var}(X) = \sigma^2$, so these parameters deserve their names.

Proposition

- If $X \sim N(\mu, \sigma^2)$, then

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1).$$

- The distribution $N(0, 1)$ is called the *standard normal distribution*.

Definition (Multivariate Normal Distribution)

- The *multivariate normal distribution* on $\mathbb{R}^n$ with mean $\mu \in \mathbb{R}^n$ and covariance matrix $\Sigma \in \mathbf{PD}(n)$ is given by the p.d.f.

$$f(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \frac{1}{\sqrt{\det \Sigma}} \exp\left[-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right], \text{ for } x \in \mathbb{R}^n.$$

- Denoted by $N(\mu, \Sigma)$.
- If $X \sim N(\mu, \Sigma)$, then $E(X) = \mu$, and $\text{cov}(X) = \Sigma$, so these parameters deserve their names.
- The random variables $X_1, \ldots, X_n$ are *jointly normal*.

Covariance and Independence

Definition

- Two random variables X and Y with $\text{cov}(X, Y) = 0$ are said to be *uncorrelated*.
- In general, if X and Y are independent, then X and Y are uncorrelated:

$$X \text{ and } Y \textit{independent} \Rightarrow \text{cov}(X, Y) = 0.$$

- The converse is generally not true. There are examples of uncorrelated random variables that are dependent.
- For jointly normal random variables, independence is equivalent to being uncorrelated.

Proposition

- Suppose $Z_1, \ldots, Z_n$ are IID $N(0, 1)$ random variables.
- Then $Z = (Z_1, \ldots, Z_n)' \sim N(0, I)$.

Proposition

- Suppose $\mu \in \mathbb{R}^n$ and Σ is an $n \times n$ non-negative definite matrix.
- Note that Σ has a non-negative definite square root $\Sigma^{\frac{1}{2}}$.
- Then $X \sim N(\mu, \Sigma)$ iff there exists a random vector $Z \sim N(0, I)$, such that

$$X = \mu + \Sigma^{\frac{1}{2}} Z.$$

Proposition

- Let $X \sim N(\mu, \Sigma)$, $A \in \mathbb{R}^{m \times n}$, and $c, d \in \mathbb{R}^n$.
- $$AX \sim N(A\mu, A\Sigma A')$$
- That is, AX has a multivariate normal distribution, and
$$E(AX) = AE(X) \text{ and } \mathrm{cov}(AX) = A\mathrm{cov}(X)A'.$$
- The covariance between $c'X$ and $d'X$ is
$$\mathrm{cov}(c'X, d'X) = c'\Sigma d.$$