Nonparametric Statistics Notes

Chapter 3: Some Tests Based on the Binomial Distribution

Jesse Crawford

Department of Mathematics
Tarleton State University

Quantiles

Definition

- Let X be a random variable and $0 \leq p \leq 1$.
- x_p is the *quantile* of order p of X if

$$P(X < x_p) \leq p, \text{ and}$$
$$P(X > x_p) \leq 1 - p.$$

- If more than one number satisfies these conditions, let x_p be the midpoint of the interval of numbers satisfying these conditions.
- Also called the $(100p)$th percentile.

Notation

The pth quantile for the $N(0, 1)$ distribution is z_p, so

$$P(Z < z_p) = p, \text{ and } P(Z > z_p) = 1 - p.$$

Outline

The Binomial Test

Example

- A machine manufactures parts.
- $p =$ probability that a part is defective
- Assume parts are statistically independent.
- Take a sample of $n = 10$ parts.
- Sample contains 4 defective parts.
- Testing problem:

$$\mathrm{H}_0 : p \leq 0.05 \text{ vs. } \mathrm{H}_1 : p > 0.05.$$

- Test statistic T
- Null distribution of T
- Decision rule/Critical region
- p-value
- Power
- Confidence intervals

The Binomial Test

Data and Assumptions

- n statistically independent trials
- Each trial results in “class 1” or “class 2”
- $p = P(\text{class } 1)$ for a single trial
- O_1 = number of observations in class 1

Hypothesis Tests

- $H_0 : p = p^\star$ vs. $H_1 : p \neq p^\star$ (Two-tailed)
- $H_0 : p \geq p^\star$ vs. $H_1 : p < p^\star$ (Lower-tailed)
- $H_0 : p \leq p^\star$ vs. $H_1 : p > p^\star$ (Upper-tailed)

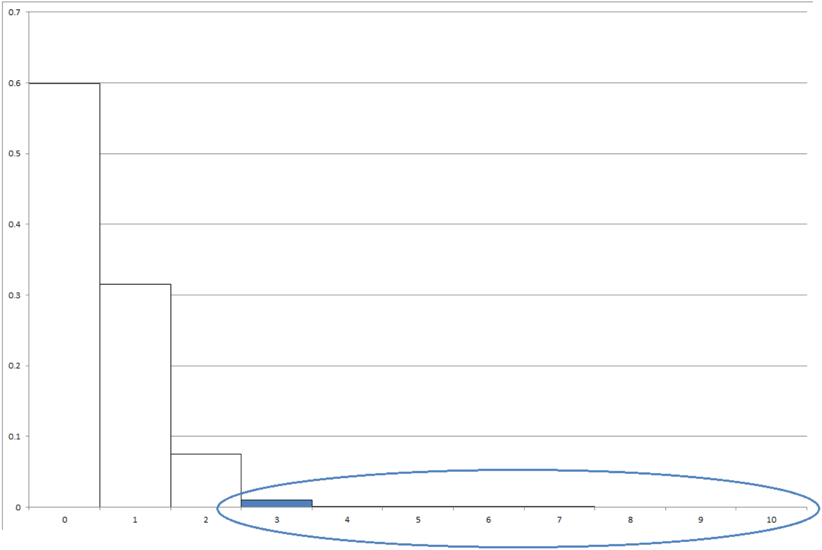
Test Statistic and Null Distribution

- Test statistic: $T = O_1$
- Null distribution: $T \sim \text{binomial}(n, p^\star)$

Upper-Tailed Binomial Test

$$H_0 : p \leq p^\star \text{ vs. } H_1 : p > p^\star$$

- Test statistic: $T = O_1$
- Null distribution: $T \sim \text{binomial}(n, p^\star)$
- Decision rule:
  - Choose t such that
    $$P(T \leq t \mid p = p^\star) \approx 1 - \alpha,$$
    (Use Table A3 or a normal approximation)
  - Reject H_0 if $T > t$
- Critical region: $[T > t]$



- binomial($n = 10, p = 0.05$)
- Significance level $= P(T > 2 \mid p = 0.05) = 1 - 0.9885 = 0.0115$

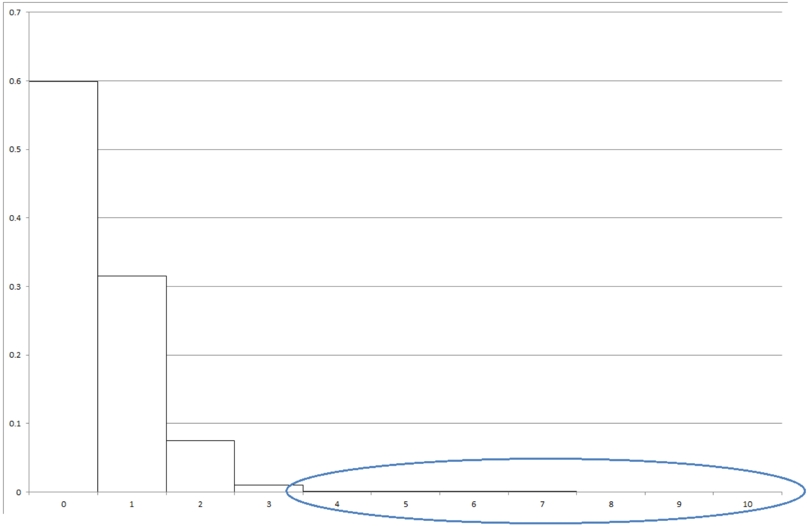
Upper-Tailed Binomial Test

$$H_0 : p \leq p^\star \text{ vs. } H_1 : p > p^\star$$

- Test statistic: $T = O_1$
- Null distribution: $T \sim \text{binomial}(n, p^\star)$
- Decision rule:
  - Choose t such that

    $$P(T \leq t \mid p = p^\star) \approx 1 - \alpha,$$

    (Use Table A3 or a normal approximation)
  - Reject H_0 if $T > t$
- Critical region: $[T > t]$
- Given $T = t_{obs}$, the p-value is $P(T \geq t_{obs} \mid p = p^\star)$.



- binomial$(n = 10, p = 0.05)$
- p-value $= P(T \geq 4 \mid p = 0.05) = 1 - 0.9990 = 0.0010$

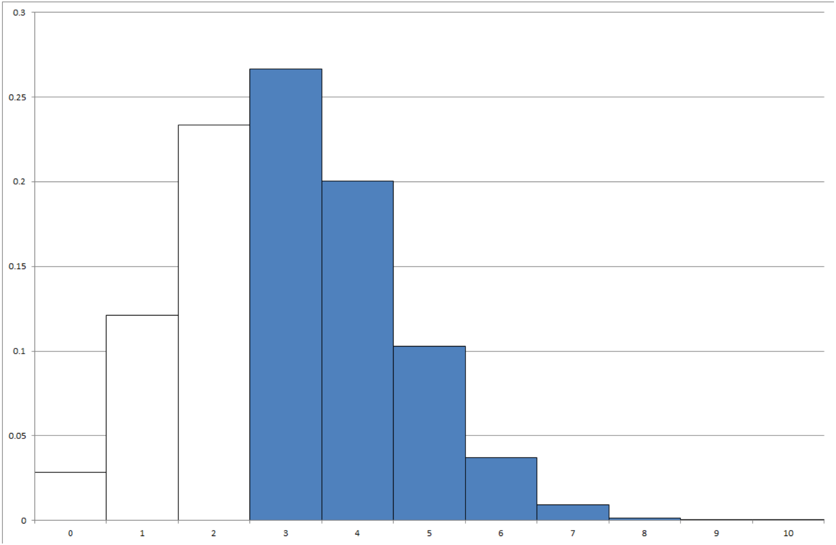
Upper-Tailed Binomial Test

$$H_0 : p \leq p^\star \text{ vs. } H_1 : p > p^\star$$

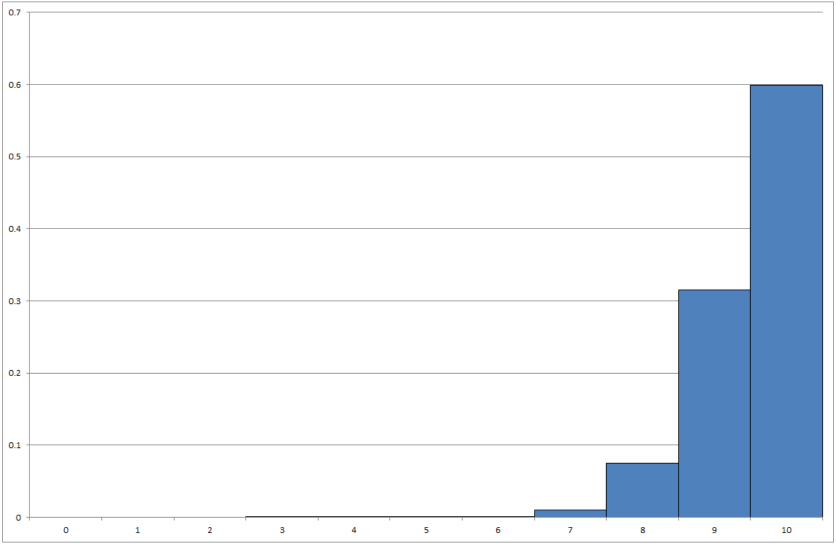
- Test statistic: $T = O_1$
- Null distribution: $T \sim \text{binomial}(n, p^\star)$
- Decision rule:
  - Choose t such that

    $$P(T \leq t \mid p = p^\star) \approx 1 - \alpha,$$

    (Use Table A3 or a normal approximation)
  - Reject H_0 if $T > t$
- Critical region: $[T > t]$
- Given $T = t_{obs}$, the p-value is $P(T \geq t_{obs} \mid p = p^\star)$.
- For any value of p, the power is $P(T > t \mid p)$



- binomial($n = 10, p = 0.3$)
- Power $= P(T > 2 \mid p = 0.3) = 1 - 0.3828 = 0.6172$



- binomial($n = 10, p = 0.95$)
- Power $= P(T > 2 \mid p = 0.95) = 1 - 0.0000 = 1.0000$

Example

- Normally, at least 50% of men undergoing a prostate cancer operation experience a certain side effect.
- New method for performing operation.
- Sample of 19 men.
- 3 experienced side effect.
- Is there statistically significant evidence that the new method has a lower chance of producing the side effect?

Lower-Tailed Binomial Test

$$H_0 : p \geq p^\star \text{ vs. } H_1 : p < p^\star$$

- Test statistic: $T = O_1$
- Null distribution: $T \sim \text{binomial}(n, p^\star)$
- Decision rule:
  - Choose t such that

    $$P(T \leq t \mid p = p^\star) \approx \alpha,$$

    (Use Table A3 or a normal approximation)
  - Reject H_0 if $T \leq t$
- Critical region: $[T \leq t]$
- Given $T = t_{obs}$, the p-value is $P(T \leq t_{obs} \mid p = p^\star)$.
- For any value of p, the power is $P(T \leq t \mid p)$

Normal Approximation for Binomial Quantiles

- Suppose $X \sim \text{binomial}(n, p)$.
- If $np \geq 5$, and $n(1-p) \geq 5$, then the qth quantile of X is approximately

$$x_q \approx np + z_q\sqrt{np(1-p)}$$

Normal Approximation for p-values

$$P(T \leq t_{\text{obs}} \mid p = p^\star) \approx P\left(Z \leq \frac{t_{\text{obs}} - np^\star + 0.5}{\sqrt{np^\star(1-p^\star)}}\right)$$

$$P(T \geq t_{\text{obs}} \mid p = p^\star) \approx P\left(Z \geq \frac{t_{\text{obs}} - np^\star - 0.5}{\sqrt{np^\star(1-p^\star)}}\right)$$

Two-Tailed Binomial Test

$$H_0 : p = p^\star \text{ vs. } H_1 : p \neq p^\star$$

- Test statistic: $T = O_1$
- Null distribution: $T \sim \text{binomial}(n, p^\star)$
- Decision rule:
  - Choose t_1 and t_2 such that

$$P(T \leq t_1 \mid p = p^\star) \approx \alpha/2$$
$$P(T \leq t_2 \mid p = p^\star) \approx 1 - \alpha/2$$

    (Use Table A3 or a normal approximation)
  - Reject H_0 if $T \leq t_1$ or $T > t_2$
- Critical region: $[T \leq t_1 \text{ or } T > t_2]$
- The p-value is $2 \cdot \min[P(T \leq t_{obs} \mid p^\star), P(T \geq t_{obs} \mid p^\star)]$.
- For any value of p, the power is $P(T \leq t_1 \text{ or } T > t_2 \mid p)$

Binomial Distribution: Confidence Interval for p

- Suppose $Y \sim \text{binomial}(n, p)$
- If $n \leq 30$ and the confidence level is 0.9, 0.95, or 0.99, the exact Clopper Pearson confidence interval is given in table A4.
- If $np \geq 5$ and $n(1-p) \geq 5$, we can use the normal approximation

$$\frac{Y}{n} \pm z_{1-\alpha/2}\sqrt{\frac{Y(n-Y)}{n^3}}$$

Note that this is the same as

$$\hat{p} \pm z_{1-\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},$$

where $\hat{p} = Y/n$.

Outline

The Quantile Test

Example

- Random sample of standardized test scores:

189	233	195	160	212
176	231	185	199	213
202	193	174	166	248

- Test whether the 75th percentile of the scores in the population is equal to 193.

$$\mathrm{H}_0 : x_{0.75} = 193 \text{ vs. } \mathrm{H}_1 : x_{0.75} \neq 193.$$

Two-Tailed Quantile Test

- Assumption: $X_1, \ldots, X_p$ is a random sample (they are IID) from a distribution whose measurement scale is at least ordinal.

$$H_0 : x_{p^\star} = x^\star \text{ vs. } H_1 : x_{p^\star} \neq x^\star$$

- Test statistics:
  - $T_1 =$ # of X_i's less than or equal to $x^\star$
  - $T_2 =$ # of X_i's less than $x^\star$
- Null distributions for both T_1 and T_2: binomial$(n, p^\star)$
- Decision rule:
  - Let Y represent a binomial$(n, p^\star)$ random variable.
  - Choose t_1 and t_2 such that

$$P(Y \leq t_1) \approx \alpha/2$$
$$P(Y \leq t_2) \approx 1 - \alpha/2$$

    (Use Table A3 or a normal approximation)
  - Reject H_0 if $T_1 \leq t_1$ or $T_2 > t_2$. Critical region: $[T_1 \leq t_1 \text{ or } T_2 > t_2]$

Example

- Random sample of standardized test scores:

189	233	195	160	212
176	231	185	199	213
202	193	174	166	248

$$H_0 : x_{0.75} = 193 \text{ vs. } H_1 : x_{0.75} \neq 193.$$

- $T_1 = \#$ of X_i's less than or equal to $x^\star$
- $T_2 = \#$ of X_i's less than $x^\star$
- Choose t_1 and t_2 such that

$$P(Y \leq t_1) \approx \alpha/2$$
$$P(Y \leq t_2) \approx 1 - \alpha/2$$

- Reject H_0 if $T_1 \leq t_1$ or $T_2 > t_2$

Lower-Tailed Quantile Test

$$H_0 : x_{p^\star} \le x^\star \text{ vs. } H_1 : x_{p^\star} > x^\star$$

- Test statistic:
  - $T_1 =$ # of X_i's less than or equal to $x^\star$
- Null distribution: $T_1 \sim \text{binomial}(n, p^\star)$
- Decision rule:
  - Let Y represent a binomial$(n, p^\star)$ random variable.
  - Choose t_1 such that

$$P(Y \le t_1) \approx \alpha$$

    (Use Table A3 or a normal approximation)
  - Reject H_0 if $T_1 \le t_1$
- Critical region: $[T_1 \le t_1]$

Upper-Tailed Quantile Test

$$H_0 : x_{p^\star} \geq x^\star \text{ vs. } H_1 : x_{p^\star} < x^\star$$

- Test statistic:
  - $T_2 = \#$ of X_i's less than $x^\star$
- Null distribution: $T_2 \sim \text{binomial}(n, p^\star)$
- Decision rule:
  - Let Y represent a binomial$(n, p^\star)$ random variable.
  - Choose t_2 such that

    $$P(Y \leq t_2) \approx 1 - \alpha$$

    (Use Table A3 or a normal approximation)
  - Reject H_0 if $T_2 > t_2$
- Critical region: $[T_2 > t_2]$

Outline

The Sign Test

Example

- 100 people tested two products.
  - 15 people preferred product A to product B
  - 4 people preferred product B to product A
  - 81 people had no preference
- Summary:
  - 15 $+$'s
  - 4 $-$'s
  - 81 ties

$$\text{H}_0 : P(+) = P(-) \text{ vs. } \text{H}_1 : P(+) \neq P(-)$$

Two-Tailed Sign Test

$$H_0 : P(+) = P(-) \text{ vs. } H_1 : P(+) \neq P(-)$$

- $n = [\# \text{ of } +\text{'s}] + [\# \text{ of } -\text{'s}]$
- Test statistic: $T = [\# \text{ of } +\text{'s}]$
- Null distribution: $T \sim \text{binomial}(n, \frac{1}{2})$
- Decision rule:
  - Let Y represent a binomial$(n, \frac{1}{2})$ random variable.
  - Choose t_1 and t_2 such that

    $$P(Y \leq t_1) \approx \alpha/2$$
    $$P(Y \leq t_2) \approx 1 - \alpha/2$$

    (Use Table A3 or a normal approximation)
  - Reject H_0 if $T \leq t_1$ or $T > t_2$
- Critical region: $[T \leq t_1 \text{ or } T > t_2]$

Outline

The McNemar Test for Significance of Changes

Example

- Presidential Debate
- Summary of voter intentions

		After: Democrat	After: Republican
Before	Democrat	63	21
	Republican	4	12

- Test whether a statistically significant difference in voter intentions exists before and after the debate.

Cox-Stuart Test for Trend

Example

- Precipitation readings for 19 years:

45.2	45.8	41.7	36.2	45.3	52.2	35.3	57.1	35.3	57.1
41.0	33.7	45.7	37.9	41.7	36.0	49.8	36.2	39.9	

- Test whether a trend in this data exists.