Nonparametric Statistics Notes

Chapters One and Two: Preliminaries

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Outline

1. Set Theory Review
2. Brief Overview of Measure Theory
3. Probability Theory
4. The Hypergeometric, Binomial, and Normal Distributions
5. Statistical Inference

Sets and Elements

Definition

- A *set* is a collection of mathematical objects, called the *elements* of the set.
- $x \in A$ means that x is an element of the set A.
- $x \notin A$ means that x is not an element of the set A.

Example

- Let $A = \{1, 2, 3\}$
  - $1 \in A$
  - $2 \in A$
  - $3 \in A$
  - $7 \notin A$
- The order that elements are listed in a set doesn't matter.

$$\{4, 8, 9, 15\} = \{8, 15, 4, 9\}$$

Example

Sets can contain infinitely many elements. If the set exhibits an obvious pattern, it may be sufficient to simply list a few elements of the set.

- Let $A = \{5, 10, 15, \ldots\}$
  - $25 \in A$
  - $28 \notin A$

Sets can also be defined with *set builder notation*.

- Let $B = \{x \mid 4 \leq x \leq 9\}$
  - $6 \in B$
  - $10 \notin B$

Here is common notation used for certain important sets of numbers.

- $\mathbb{N} = \{1, 2, 3, \ldots\}$ (Set of Natural Numbers)
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (Set of Integers)
- $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$ (Set of Rational Numbers)
- $\mathbb{R}$ (Set of Real Numbers)
- $\mathbb{C}$ (Set of Complex Numbers)

Interval Notation

- $(a, b) = \{x \in \mathbb{R} | a < x < b\}$
- $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$
- $[a, b) = \{x \in \mathbb{R} | a \leq x < b\}$
- $(a, b] = \{x \in \mathbb{R} | a < x \leq b\}$
- $(a, \infty) = \{x \in \mathbb{R} | a < x\}$
- $[a, \infty) = \{x \in \mathbb{R} | a \leq x\}$
- $(-\infty, b) = \{x \in \mathbb{R} | x < b\}$
- $(-\infty, b] = \{x \in \mathbb{R} | x \leq b\}$
- $(-\infty, \infty) = \mathbb{R}$

Unions

Definition

Given two sets A and B, the *union* of A and B is

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Note: In mathematical logic, “or” means “and/or”, so “$x \in A$ or $x \in B$” means $x \in A$ or $x \in B$ or both.

Example

- If $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6, 7\}$, then

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}.$$

- If $A = [0, 8)$ and $B = (6, 10]$, then

$$A \cup B = [0, 10].$$

Intersections

Definition

Given two sets A and B, the *intersection* of A and B is

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Example

- If $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6, 7\}$, then
$$A \cap B = \{3, 4, 5\}.$$
- If $A = [0, 8)$ and $B = (6, 10]$, then
$$A \cap B = (6, 8).$$

Universal Sets and Complements

Definition

- In most situations, there is a *universal set* U that contains all elements under consideration.
- The *complement* of a set A is

$$A' = \{x \in U \mid x \notin A\}.$$

Example

- If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1, 2, 3\}$, then

$$A' = \{4, 5, 6, 7, 8, 9, 10\}.$$

- If $U = \mathbb{R}$ and $A = [5, 8)$, then

$$A' = (-\infty, 5) \cup [8, \infty).$$

Set Difference

Definition

Given two sets A and B,

$$A \setminus B = \{x \in A \mid x \notin B\}.$$

Example

- If $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6, 7\}$, then

$$A \setminus B = \{1, 2\}.$$

- If $A = [0, 8)$ and $B = (6, 10]$, then

$$A \setminus B = [0, 6].$$

Note that complements are just a special case of set difference, since

$$A' = \{x \in U \mid x \notin A\} = U \setminus A.$$

Subsets and the Empty Set

Definition

A is a subset of B, written $A \subseteq B$, if every element of A is an element of B.

Example

- If $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$, then $A \subseteq B$.
- If $A = \{1, 2, 3\}$, and $B = \{1, 2, 8, 9\}$, then $A \not\subseteq B$, because $3 \in A$ but $3 \notin B$.

Definition

The *empty set* is the set $\emptyset = \{\}$ that contains no elements.

- For any set A
  - $A \cup \emptyset = A$
  - $A \cap \emptyset = \emptyset$
  - $\emptyset \subseteq A$
  - $A \subseteq A$
  - $A \subseteq U$, where U is the universal set.

Families of Sets

Definition

- A *family* $\mathcal{F}$ of sets is simply a set whose elements are also sets.
- The union of all sets in the family is

$$\bigcup_{A \in \mathcal{F}} A = \{x \mid x \in A, \text{ for some } A \in \mathcal{F}\}.$$

- The intersection of all sets in the family is

$$\bigcap_{A \in \mathcal{F}} A = \{x \mid x \in A, \text{ for all } A \in \mathcal{F}\}.$$

Example

If $\mathcal{F} = \{[-x, x] \mid x > 0\}$, evaluate the following

$$\bigcup_{A \in \mathcal{F}} A \text{ and } \bigcap_{A \in \mathcal{F}} A.$$

Definition

- If A_i is a set, for every $i \in I$, then
- $$\bigcup_{i \in I} A_i = \{x \mid x \in A_i, \text{ for some } i \in I\}.$$
  $$\bigcap_{i \in I} A_i = \{x \mid x \in A_i, \text{ for all } i \in I\}.$$
- The variable i is called the *index*, and I is the *index set*. When the index set is $I = \{1, 2, 3, \ldots\}$, we can write
  $$\bigcup_{i=1}^{\infty} A_i \text{ and } \bigcap_{i=1}^{\infty} A_i.$$

Example

$$\bigcap_{i=1}^{\infty} (0, 5 + \tfrac{1}{i})$$

The Power Set of a Set

Definition

- Let U be a set.
- The *power set* of U, denoted $\mathcal{P}(U)$, is the set of all subsets of U.

$$\mathcal{P}(U) = \{A \mid A \subseteq U\}.$$

- Note that
  - $A \subseteq U$ if and only if $A \in \mathcal{P}(U)$.
  - $\mathcal{F}$ is a family of subsets of U if and only if $\mathcal{F} \subseteq \mathcal{P}(U)$.

Example

If $U = \{1, 2, 3\}$, then

- $\mathcal{P}(U) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, U\}$.
- $\{1, 3\} \subseteq U$, so $\{1, 3\} \in \mathcal{P}(U)$.
- If $\mathcal{F} = \{\{x\} \mid x \in U\} = \{\{1\}, \{2\}, \{3\}\}$, then $\mathcal{F} \subseteq \mathcal{P}(U)$.

Cartesian Products and Functions

Definition

Given two sets U and V, the *cartesian product* of U and V is the following set of ordered pairs:

$$U \times V = \{(u, v) \mid u \in U \text{ and } v \in V\}.$$

Definition

A *function* f from U to V is a subset of $U \times V$, such that for any $u \in U$, there exists a unique $v \in V$, such that $(u, v) \in f$.

- Functions are often called maps or mappings.
- U and V are the *domain* and *codomain* of f, respectively.
- To indicate that f is a function from U to V, we write $f : U \to V$.
- If $(u, v) \in f$, we write $f(u) = v$.

Images and Inverse Images

Definition

Let $f : U \to V$, and let $A \subseteq U$ and $B \subseteq V$.

- The *image* of A under f is
$$f(A) = \{f(x) \mid x \in A\}.$$
- The *inverse image* of B under f is
$$f^{-1}(B) = \{x \in U \mid f(x) \in B\}.$$
- The *image* or *range* of f is $f(U) = \{f(x) \mid x \in U\}$.

Example

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Then

- $f([0,5]) = [0,25]$
- $f^{-1}([0,25]) = [-5,5]$
- The range of f is $[0,\infty)$.

Injective and Surjective Functions

Definition

Let $f : U \to V$ then

- f is *injective* (also called one-to-one) if for all $u_1, u_2 \in U$,

$$f(u_1) = f(u_2) \text{ implies } u_1 = u_2.$$

- f is *surjective* (also called onto) if $f(U) = V$, that is,

$$\text{for any } v \in V, \text{ there exists } u \in U, \text{ such that } f(u) = v.$$

- A function that is both injective and surjective is called bijective.

Example

Give examples of injective and surjective functions.

Countable and Uncountable Sets

Definition

- A set A is *countably infinite* if there is a bijection between A and the set of natural numbers $\mathbb{N}$.
- A set A is *countable* if it is finite or countably infinite.
- A set is *uncountable* if it is not countable.

The following are equivalent:

- A is countable
- There exists an injective function $f : A \to \mathbb{N}$
- There exists a surjective function $f : \mathbb{N} \to A$.

Example

Which of the following sets are countable?
$\{0, 1, 2, \ldots, 10\}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

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σ-algebras

Definition

- Suppose Ω is a set and $\mathcal{F}$ is a family of subsets of Ω.
- Then $\mathcal{F}$ is called a σ-algebra (or σ-field) if the following conditions hold:
  - $\Omega \in \mathcal{F}$
  - $A \in \mathcal{F}$ implies $A' \in \mathcal{F}$
  - $A_1, A_2, \ldots \in \mathcal{F}$ implies $A_1 \cup A_2 \cup \cdots \in \mathcal{F}$.
- Elements of $\mathcal{F}$ are called *measurable sets*.
- $(\Omega, \mathcal{F})$ is called a *measurable space*.

Example

Show that a σ-algebra always has these two properties also:

- $\emptyset \in \mathcal{F}$
- $A_1, A_2, \ldots \in \mathcal{F}$ implies $A_1 \cap A_2 \cap \cdots \in \mathcal{F}$.

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Example

- Let Ω be any set.
- Prove that $\mathcal{P}(\Omega)$, the power set of Ω, is a σ-algebra.

σ-algebras

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Example

- Let $\Omega = \mathbb{R}$. There exits a σ-algebra $\mathcal{B}$, called the Borel σ-algebra, such that $(a, b) \in \mathcal{B}$, for all $a, b \in \mathbb{R}$.
- Prove that $[a, b] \in \mathcal{B}$, $\{a\} \in \mathcal{B}$, and $(a, \infty) \in \mathcal{B}$, for all $a, b \in \mathbb{R}$.

Pairwise Disjoint Sets

Definition

If $A \cap B = \emptyset$, then A and B are *disjoint*.

Definition

- A family of sets A_i, $i \in I$ is called *pairwise disjoint* if

$$A_i \cap A_j = \emptyset, \text{ if } i \neq j.$$

- Also called *mutually exclusive*.

Measures

Definition

- Suppose $(\Omega, \mathcal{F})$ is a measurable space.
- A function $\mu : \mathcal{F} \to [0, \infty]$ is called a *measure* on $(\Omega, \mathcal{F})$ if
  - $\mu(\emptyset) = 0$
  - For any sequence of pairwise disjoint sets $A_1, A_2, \ldots \in \mathcal{F}$,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

- If μ is a measure on $(\Omega, \mathcal{F})$, then $(\Omega, \mathcal{F}, \mu)$ is called a *measure space*.

Example

- Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $A, B \in \mathcal{F}$.
- If $A \subseteq B$, prove $\mu(A) \leq \mu(B)$.
- If $A \subseteq B$, and $\mu(B) < \infty$, prove that $\mu(B \setminus A) = \mu(B) - \mu(A)$.

- A function $\mu : \mathcal{F} \to [0, \infty]$ is called a *measure* on $(\Omega, \mathcal{F})$ if
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$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Example

- Let $(\Omega, \mathcal{F}) = (\mathbb{R}, \mathcal{B})$. There exists a measure μ on this space called *Lebesgue measure*, such that

$$\mu((a, b)) = b - a, \text{ for all real numbers } a < b.$$

- For all real numbers $a < b$, prove the following:
  - $\mu(\{a\}) = 0$
  - $\mu([a, b]) = b - a$
  - $\mu((0, \infty)) = \infty$
  - $\mu(\mathbb{R}) = \infty$.

Measurable Functions

Definition

- Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces.
- A function $f : \Omega_1 \to \Omega_2$ is *measurable* if

$$f^{-1}(A) \in \mathcal{F}_1, \text{ for any } A \in \mathcal{F}_2.$$

Example

- Let $(\Omega_1, \mathcal{F}_1) = (\mathbb{R}, \mathcal{B})$, and let $(\Omega_2, \mathcal{F}_2) = (\mathbb{Z}, \mathcal{P}(\mathbb{Z}))$.
- Define $f : \mathbb{R} \to \mathbb{Z}$ by $f(x) = \lfloor x \rfloor$, the greatest integer less than or equal to x.
- Prove that f is measurable.

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Probability Measures

Definition

- A *probability space* is a measure space $(S, \mathcal{F}, P)$, such that $P(S) = 1$.
- S is the *sample space* or *observation space*.
- P is called a *probability measure*.
- A set $A \in \mathcal{F}$ is called an *event*.
- $P(A)$ is the probability of the event A.

- $P(\emptyset) = 0$
- $P(S) = 1$
- For all $A, B \in \mathcal{F}$
    - If $A \subseteq B$, then $P(A) \leq P(B)$.
    - $0 \leq P(A) \leq 1$
    - $P(A') = 1 - P(A)$
    - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Probability Measures

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- A set $A \in \mathcal{F}$ is called an *event*.
- $P(A)$ is the probability of the event A.

Example

- $S = \{1, 2, 3, \ldots\}$
- $\mathcal{F} = \mathcal{P}(S)$
- Assume $P(\{s\}) = 2^{-s}$, for all $s \in S$
- Find these probabilities:
  - $P(\{1, 2\})$
  - $P(\{3, 4, 5, \ldots\})$

Discrete Probability Distributions

Definition

- Suppose $(S, \mathcal{P}(S), P)$ is a probability space.
- If S is a countable set, then P is a *discrete* probability measure.
- P is completely determined by the function

$$\begin{aligned} f : S &\to [0,1] \\ f(s) &= P(\{s\}). \end{aligned}$$

- For any $A \subseteq S$,

$$P(A) = \sum_{s \in A} f(s).$$

- f is called the *probability mass function* (p.m.f.) or *probability distribution function* (p.d.f.) of P.
- Note that
  - $0 \leq f(s) \leq 1$, for all $s \in S$
  - $\sum_{s \in S} f(s) = 1$.

Random Variables

Definition

- Let $(S, \mathcal{F}, P)$ be a probability space.
- A *random variable* is a measurable function

$$X : S \to \mathbb{R}.$$

- ($\mathbb{R}$ is equipped with the Borel σ-algebra $\mathcal{B}$.)

Example

$$X : S \to \mathbb{R}$$

$$X(s) = \begin{cases} 0 & \text{if } s = TTT \\ 1 & \text{if } s \in \{HTT, THT, TTH\} \\ 2 & \text{if } s \in \{HHT, HTH, THH\} \\ 3 & \text{if } s = HHH \end{cases}$$

The Distribution of a Random Variable

Notation

If $X : S \to \mathbb{R}$ is a random variable,

$$[X \in A] = \{s \in S \mid X(s) \in A\}$$

Definition

- Suppose $(S, \mathcal{F}, P)$ is a probability space.
- Let $X : S \to \mathbb{R}$ be a random variable.
- Then there is a corresponding probability space $(\mathbb{R}, \mathcal{B}, P_X)$, where

$$P_X(A) = P[X \in A] = P[X^{-1}(A)], \text{ for all } A \in \mathcal{B}.$$

- P_X is the *probability distribution* of X.
- It's often just called the *distribution* of X.

Support of a Random Variable

Definition

If $X : S \to \mathbb{R}$ is a random variable, the *support* of X is the set of all possible values of X:

$$\text{supp}(X) = \{X(s) \mid s \in S\} = X(S).$$

Definition

- A random variable X is *discrete* if $\text{supp}(X)$ is countable.
- If X is discrete, the distribution of X is completely determined by the function

$$f : \mathbb{R} \to [0, 1]$$
$$f(x) = P_X(\{x\}) = P[X = x].$$

- f is called the *probability mass function* (p.m.f.) or *probability distribution function* (p.d.f.) of X.

Discrete Random Variables

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- Note that
  - $0 \leq f(x) \leq 1$, for all $x \in \mathbb{R}$
  - $\sum_{x \in \mathbb{R}} f(x) = 1$.

Expected Value and Variance of a Random Variable

Definition

- Let X be a discrete random variable with p.m.f. f.
- The *expected value* of X is

$$\mu = \mu_X = E(X) = \sum_{x \in \mathbb{R}} x f(x).$$

- Given any function $u : \mathbb{R} \to \mathbb{R}$,

$$E[u(X)] = \sum_{x \in \mathbb{R}} u(x) f(x)$$

- The *variance* of X is

$$\sigma^2 = \sigma_X^2 = \text{Var}(X) = E[(X - \mu_X)^2] = E(X^2) - E(X)^2.$$

- The *standard deviation* of X is $\sigma = \sigma_X = \sqrt{\text{Var}(X)}$.

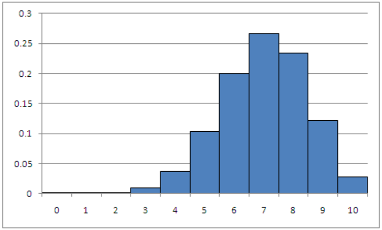

Figure: Binomial distribution with $n = 10$ and $p = 0.7$.

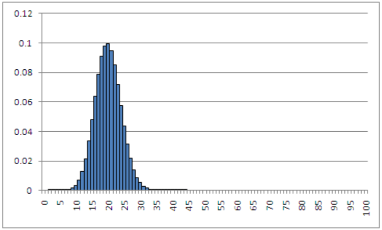

Figure: Binomial distribution with $n = 100$ and $p = 0.2$.

Moment Generating Function of a Random Variable

Definition

- Let X be a random variable.
- Assume there exists $h > 0$, such that

$$M(t) = E[e^{tX}] \text{ converges, for } -h < t < h.$$

- Then M is called the *moment-generating function* (m.g.f.) of X.
- If the above expected value does not exists on some interval $(-h, h)$, then the m.g.f. does not exist.

If X is a random variable, and its m.g.f. exists, then

$$E(X^r) = M^{(r)}(0), \text{ for any } r = 1, 2, \ldots$$

Cumulative Distribution Function of a Random Variable

Definition

- The *cumulative distribution function* (c.d.f.) of the random variable X is

$$F : \mathbb{R} \to [0, 1]$$
$$F(x) = P[X \leq x].$$

- Also called the distribution function.

- Let X be a discrete random variable.
- The *distribution* or *probability distribution* of X is the probability measure

$$P_X : \mathcal{B} \to [0,1]$$
$$P_X(A) = P[X \in A]$$

- The *probability mass function* (p.m.f.) or *probability distribution function* (p.d.f.) of X is

$$f : \mathbb{R} \to [0,1]$$
$$f(x) = P[X = x]$$

- The *cumulative distribution function* (c.d.f.) or *distribution function* (d.f.) of the random variable X is

$$F : \mathbb{R} \to [0,1]$$
$$F(x) = P[X \leq x].$$

Identically Distributed Random Variables

Definition

- Let X and Y be two random variables.
- Then X and Y have the same distribution if

$$P_X = P_Y$$

- We say that X and Y are *identically distributed*.

Given two discrete random variables X and Y, the following are equivalent:

- X and Y are identically distributed, $P_X = P_Y$
- X and Y have the same p.m.f., $f_X = f_Y$
- X and Y have the same c.d.f., $F_X = F_Y$
- X and Y have the same m.g.f., $M_X = M_Y$

Continuous Random Variables

Definition

- Let X be a random variable, and suppose there is a function

$$f : \mathbb{R} \to [0, \infty)$$

  such that, for any real numbers $a < b$,

$$P(a < X < b) = \int_a^b f(x)\, dx.$$

- Then the distribution of X is *continuous*, and f is the *probability density function* (p.d.f.).
- For any $A \in \mathcal{B}$,

$$P(X \in A) = \int_A f(x)\, dx.$$

  - $0 \leq f(x)$, for all $x \in \mathbb{R}$
  - $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

Discrete vs. Continuous Random Variables

- Discrete
- f is the probability mass function (p.m.f.)
- $\sum_{x \in \mathbb{R}} f(x) = 1$
- $P(X \in A) = \sum_{x \in A} f(x)$
- $E[u(X)] = \sum_{x \in \mathbb{R}} u(x) f(x)$

- Continuous
- f is the probability density function (p.d.f.)
- $\int_{\mathbb{R}} f(x)\, dx = 1$
- $P(X \in A) = \int_A f(x)\, dx$
- $E[u(X)] = \int_{\mathbb{R}} u(x) f(x)\, dx$

Given two continuous random variables X and Y, the following are equivalent:

- X and Y are identically distributed, $P_X = P_Y$
- The p.d.f.'s of X and Y are equal *almost everywhere*,

$$\mu(\{x \mid f_X(x) \neq f_Y(x)\}) = 0.$$

- X and Y have the same c.d.f., $F_X = F_Y$
- X and Y have the same m.g.f., $M_X = M_Y$

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Combinations

- Consider a set of n objects.
- The number of different subsets of size r is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Example

How many five card poker hands are there?

Example

How many distinct permutations are there of the letters

$$AAAAAAFFF$$

The Hypergeometric Distribution

Example

- A car dealership has 20 cars: 12 Fords and 8 Chevrolets.
- 5 cars are selected at random without replacement.
- What's the probability of selecting exactly 3 Fords?

Hypergeometric Distribution

- N_1 = number of objects of type 1
- N_2 = number of objects of type 2
- Random sample without replacement.
- n = sample size.
- X = number of objects in sample of type 1

$$P(X = x) = f(x) = \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N_1+N_2}{n}}, \text{ for } x \leq n, x \leq N_1, n - x \leq N_2.$$

Independent Events

Definition

Two events A and B are *statistically independent* if

$$P(A \cap B) = P(A)P(B).$$

Example

- $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
- All outcomes are equilikely.
- $A = \{HHH, HHT, HTH, HTT\}$ (First coin is heads)
- $B = \{HHH, HHT, THH, THT\}$ (Second coin is heads)

Independence of Multiple Events

Definition

A family of events $\{A_i \mid i \in I\}$, is statistically independent if

$$P(A_{i_1} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) \cdots P(A_{i_k}),$$

for any subfamily of events $\{A_{i_1}, \ldots, A_{i_k}\}$.

Independent Random Variables

Definition

Two random variables X and Y are *statistically independent* if

$$P(X \in A \text{ and } Y \in B) = P(X \in A)P(Y \in B), \text{ for all } A, B \in \mathcal{B}$$

The Binomial Distribution

Example

- A football player kicks 10 field goals.
- Chance of making each field goal is 80%.
- The field goals are statistically independent.
- Find the probability of making exactly 7 of the field goals.

Binomial Distribution

- Sequence of n trials
- Each trial has only two possible outcomes, “success” and “failure”
- $p =$ probability of “success” on a single trial
- Trials are statistically independent
- $X =$ number of “successes” that actually occur

$$P(X = x) = f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \text{ for } x = 0, 1, \ldots, n.$$

Example

Find the probability of making between 6 and 9 field goals inclusive?

The Normal Distribution

Definition

- Suppose $\mu \in \mathbb{R}$, and $\sigma^2 > 0$.
- The *normal distribution* with mean μ and variance σ^2, is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \ -\infty < x < \infty.$$

Notation: $N(\mu, \sigma^2)$

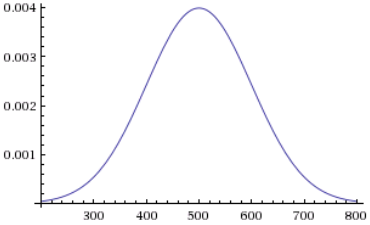


Figure: Normal distribution with $\mu = 500$ and $\sigma = 100$.

Standard Normal Distribution

Definition

The distribution $N(\mu = 0, \sigma^2 = 1)$ is the *standard normal distribution*.

Standardizing a Normal Random Variable

If $X \sim N(\mu, \sigma^2)$, then

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1).$$

The Central Limit Theorem

Theorem (5.6-1)

- *Suppose $X_1, X_2, \ldots$ is a sequence of IID random variables,*
- *from a distribution with finite mean μ*
- *and finite positive variance σ^2.*
- *Let $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, for $n = 1, 2, \ldots$*
- *Then, as $n \to \infty$,*

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma} \Rightarrow N(0, 1).$$

Informal Statement of CLT

Informal CLT

- Suppose $X_1, \ldots, X_n$ is a random sample
- from a distribution with finite mean μ
- and finite positive variance σ^2.
- Then, if n is sufficiently large,

$$\overline{X} \approx N(\mu, \sigma^2/n), \text{ and}$$

$$\sum_{i=1}^{n} X_i \approx N(n\mu, n\sigma^2).$$

- Conventionally, values of $n \geq 30$ are usually considered sufficiently large, although this text applies the approximation for lower values of n, such as $n \geq 20$.

Normal Approximation to the Binomial Distribution

If $np \geq 5$ and $n(1-p) \geq 5$, then

$$b(n,p) \approx N(\mu = np, \sigma^2 = np(1-p)).$$

Outline

1 Set Theory Review

2 Brief Overview of Measure Theory

3 Probability Theory

4 The Hypergeometric, Binomial, and Normal Distributions

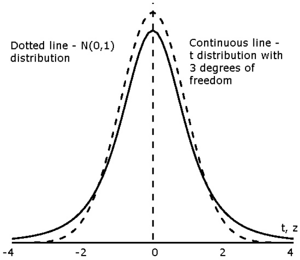
5 Statistical Inference

Definition

- Let r be a positive integer.
- The t-distribution with r degrees of freedom is given by

$$f(t) = \frac{\Gamma((r+1)/2)}{\sqrt{\pi r}\Gamma(r/2)} \frac{1}{(1+t^2/r)^{(r+1)/2}}, \quad -\infty < t < \infty.$$

- As $r \to \infty$, $t(r) \Rightarrow N(0,1)$.



Test Statistics for the Normal Distribution

Proposition

- Consider a random sample $X_1, \ldots, X_n$ from a $N(\mu, \sigma^2)$ population.
- $X_1, \ldots, X_n$ are IID, and $X_i \sim N(\mu, \sigma^2)$, for all i.
- $$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$
- $$T = \frac{\overline{X} - \mu}{S/\sqrt{n}} \sim t(n-1).$$

Critical Values for Normal and t-distributions

Definition

- Let $Z \sim N(0,1)$ and $T \sim t(r)$.
- Let $\alpha \in (0,1)$.
- We define z_α and $t_\alpha(r)$ as follows:

$$P[Z > z_\alpha] = \alpha.$$

$$P[T > t_\alpha(r)] = \alpha.$$

α	$z_{\alpha/2}$	$t_{\alpha/2}(30)$
0.10	1.645	1.697
0.05	1.96	2.042
0.01	2.575	2.750

A Hypothesis Testing Example

Example

- Assume that packages of M&M's have a $N(\mu, \sigma^2)$ distribution.
- A sample of 31 packages of M&M's had sample mean $\overline{X} = 235.1$ grams and sample standard deviation $S = 5.7$ grams.
- Perform the following hypothesis test at the $\alpha = 0.05$ significance level

$$H_0 : \mu = 232.5 \text{ vs. } H_1 : \mu \neq 232.5.$$

Hypothesis Test: Mean of a Normal Distribution

- Let $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$ be a random sample.
- For the testing problem

$$\mathrm{H}_0 : \mu = \mu_0 \text{ vs. } \mathrm{H}_1 : \mu \neq \mu_0,$$

- the **test statistic** is

$$T = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}.$$

- The **null distribution** of T is

$$T \sim t(n-1).$$

- The **decision rule** is

$$\begin{aligned} &\text{Reject } \mathrm{H}_0\text{, if } |T| \geq t_{\alpha/2}(n-1) \\ &\text{Do not reject } \mathrm{H}_0\text{, otherwise.} \end{aligned}$$

- The **critical region** is $C = \{t \in \mathbb{R} \mid |t| \geq t_{\alpha/2}(n-1)\}$.

Hypothesis Testing Terminology

Definition

- A **statistic** is any variable computed based on a sample of data.
- A **test statistic** is a statistic used to perform a hypothesis test.
- The **null distribution** of a test statistic is its distribution under the assumption that H_0 is true.
- The **critical region** or **rejection region** of a test is the set of all values of the test statistic that result in rejecting H_0.

Two Types of Errors

- Type I error: Rejecting H_0 when it is true.
- Type II error: Not rejecting H_0 when it is false.

Given the Null Hypothesis Is

Your Decision Based On a Random Sample		True	False
	Reject	Type I Error	Correct Decision
	Do Not Reject	Correct Decision	Type II Error

Two Types of Errors in Decision Making

- The **significance level** of a test is

$$\alpha = \max\{P(\text{type I error}) \mid H_0 \text{ is true}\}.$$

Given the Null Hypothesis Is

Your Decision Based On a Random Sample		True	False
	Reject	Type I Error	Correct Decision
	Do Not Reject	Correct Decision	Type II Error

Two Types of Errors in Decision Making

- The **significance level** of a test is

$$
\begin{aligned}
\alpha &= \max\{P(\text{type I error}) \mid \text{H}_0 \text{ is true}\} \\
&= \max\{P(\text{rejecting H}_0) \mid \text{H}_0 \text{ is true}\} \\
&= \max\{P(T \in C) \mid \text{H}_0 \text{ is true}\}
\end{aligned}
$$

p-values

Definition

- The *p*-**value** is the smallest significance level at which the null hypothesis would be rejected for a given observation.
- Also called the **observed significance level.**
- It is the probability of all values more extreme than T under the null distribution.
- The smaller the *p*-value is, the stronger the evidence is **against** H_0.

Confidence Intervals

Definition

A $1-\alpha$ **confidence interval** for a parameter θ is a random interval $[L, U]$, such that

$$P(L \le \theta \le U) = 1-\alpha.$$

- Let $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$ be a random sample.
- A $1-\alpha$ confidence interval for μ is

$$\left[\overline{X} - t_{\alpha/2}\frac{S}{\sqrt{n}}, \overline{X} + t_{\alpha/2}\frac{S}{\sqrt{n}}\right]$$

- Also written as

$$\overline{X} \pm t_{\alpha/2}\frac{S}{\sqrt{n}}$$

Hypothesis Testing Conclusions

- Rejecting H_0 means there is strong evidence that H_0 is false.
- Not rejecting H_0 merely means there is a lack of strong evidence against H_0.
  There is not strong evidence in favor of anything, including H_0.