Math 3320 Foundations of Mathematics

Chapter 1: Fundamentals

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Outline

Is the following statement always true?

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b \lim_{n\to\infty} f_n(x)\, dx$$

Example

For $n = 1, 2, 3, \ldots$, define

$$f_n(x) = \begin{cases} n & \text{, if } 0 \le x \le \frac{1}{n} \\ 0 & \text{, otherwise.} \end{cases}$$

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx \ne \int_0^1 \lim_{n\to\infty} f_n(x)\, dx$$

The above example is a *counterexample* to the statement above.

Lebesgue's Dominated Convergence Theorem

- Let $\{f_n\}$ be a sequence of real-valued measurable functions on a measure space (S, Σ, μ).
- Suppose that the sequence converges pointwise to a function f and is dominated by some integrable function g in the sense that

$$|f_n(x)| \leq g(x),$$

  for all numbers n in the index set and all points $x \in S$.
- Then f is integrable, and

$$\lim_{n\to\infty} \int_S f_n \, d\mu = \int_S \lim_{n\to\infty} f_n \, d\mu.$$

Goal of this Course

- Transform from a "symbol pushing" student to one who understands foundations of mathematics.
- You will be able to understand and prove theorems like this one!

Foundations Overview

- Cornerstones of mathematics: definition, theorem, and proof.
- Mathematical concepts must be precisely **defined**.
- **Theorems** are statements about these concepts.
    - $2+2=4$
    - $\frac{d}{dx}\sin(x)=\cos(x)$
    - “Two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.”
    - Lebesgue’s Dominated Convergence Theorem
- Theorems must be **proved** according to sound logic.

Set Theory and Logic

- Set Theory:
  - $\{1, 2, 3\}$
  - $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
  - $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}$
  - $\mathbb{R} = \{\text{All real numbers}\}$
  - $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$
- Logic
  - Boolean operators: and, or, not, if-then, if and only if, implies.
  - Words used to structure proofs: Let, assume, suppose, therefore, then
  - Logical Quantifiers:
    - Universal: for all, for every, $\forall$
    - Existential: there exists, for some, $\exists$

Example

Compare these two statements:

- For all $x \in \mathbb{R}$, $x^2 = 9$.
- There exists $x \in \mathbb{R}$, such that $x^2 = 9$.

Lebesgue's Dominated Convergence Theorem

- Let $\{f_n\}$ be a sequence of real-valued measurable functions on a measure space (S, Σ, μ).
- Suppose that the sequence converges pointwise to a function f and is dominated by some integrable function g in the sense that

$$|f_n(x)| \leq g(x),$$

  for all numbers n in the index set and all points $x \in S$.
- Then f is integrable, and

$$\lim_{n\to\infty} \int_S f_n \, d\mu = \int_S \lim_{n\to\infty} f_n \, d\mu.$$

Example

What is $\lim_{x \to 0} x \sin(\frac{1}{x})$?

Definition

Let f be a function defined on some interval containing a, except possibly at a itself. Then we write

$$\lim_{x \to a} f(x) = L,$$

if, for every $\varepsilon > 0$, there exists $\delta > 0$, such that

$$0 < |x - a| < \delta \text{ implies } |f(x) - L| < \varepsilon.$$

Theorem

$\lim_{x \to 0} x \sin(\frac{1}{x}) = 0$

A Misconception About Proofs

- Misconception: Proofs are just about formatting the text with a specific style, because my teacher is picky!
- Reality:
  - As with any writing, it's important to be professional, but the chosen presentation format is not that big of a deal.
  - However, changing a **single word** in a proof can be extremely important, because it can change the meaning of that sentence and cause the proof to be **logically incorrect**.

Unique Features of Mathematics

- You don't rely on experiments or third party accounts in math. You can prove/disprove things.
- You don't have to take someone else's word for it.
- You can obtain (close to) certain knowledge.
- There really is a right answer, and you can determine what it is.
- Mathematics is the foundation for all science and technology, so we need logically sound methods for deriving mathematical knowledge.
- Math is a fun, puzzle-solving activity.

Outline

Speaking and Writing Mathematics

- Precision is a top priority. We want to avoid being vague or unclear.
- Complete Sentences
  - **Bad:** $3x + 5$
  - **Good:** When we substitute $x = -5/3$ into $3x + 5$, the result is 0.
- Mismatch of Categories
  - "Air Force One is the President of the United States."
  - **Bad:** "If the legs of a right triangle T have lengths 5 and 12, then $T = 30$."
  - **Good:** "If the legs of a right triangle T have lengths 5 and 12, then the area of T is 30."
- Avoid Pronouns
  - **Bad:** "If we move everything over, then it simplifies and that's our answer."
  - **Good:** "When we move all terms involving x to the left in Equation (12), we find those terms cancel, and that enables us to determine the value of y."

Speaking and Writing Mathematics

- Rewrite your proofs
- Learn Latex

Outline

Definition (Even)

An integer is called *even* provided it is divisible by two.

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An integer is called *even* provided it is divisible by two.

- For this definition to make sense, we need to define the terms in red.
- That would require us to define even more terms.

$$\vdots$$

- Eventually we hit the foundation: **Set Theory** (Chapter 2)

Our Starting Point: The Integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

You may use the following freely in any proof:

- Algebraic properties of addition, subtraction, and multiplication (not division).
- Basic number facts like $3 \times 2 = 6$.
- Basic facts about the order relations $(<, \leq, >, \geq)$.
- For specific details, see Appendix D.

Definition (Even)

An integer is called *even* provided it is divisible by two.

Definition (Divisible)

- Let a and b be integers.
- We say that a is *divisible* by b provided there is an integer c, such that $bc = a$.
- We also say that b *divides* a, or b is a *factor* of a, or b is a *divisor* of a.
- The notation for this is $b|a$.

Definition (Even)

An integer is called *even* provided it is divisible by two.

Definition (Odd)

An integer a is called *odd* provided there is an integer x, such that $a = 2x + 1$.

Definition (Prime)

An integer p is *prime* provided that $p > 1$ and the only positive divisors of p are 1 and p.

Definition (Composite)

A positive integer a is *composite* provided there is an integer b, such that $1 < b < a$, and $b|a$.

General Form of a Definition

An object X is called the *term being defined* provided it satisfies *specific conditions*.

Homework

- **To Turn In:** p. 6 (1–7, 9, 12, 13a)
- **To Discuss:** p. 6 (1cefg, 2, 3ce, 4, 9, 12abc)

Outline

A *theorem* is a declarative statement about mathematics for which there is a proof.

- Declarative: Not a command, not a question.
- Theorems are true.

The Pythagorean Theorem

If a and b are the lengths of the legs of a right triangle, and c is the length of the hypotenuse, then

$$a^2 + b^2 = c^2$$

Other Names for Theorems

- **Fact:** $6 + 3 = 9$
- **Proposition/Result:** A minor theorem
- **Lemma:** Theorem primarily used to prove another theorem
- **Corollary:** Theorem that follows immediately from another
- **Claim:** Theorem often used inside of the proof of another theorem

A False Statement

For any real number x,

$$\sqrt{x^2} = x.$$

A Nonsense Statement

The square root of a triangle is a circle.

A *conjecture* is a statement about mathematics whose truth is unknown.

Goldbach's Conjecture

Every even integer greater than 2 can be expressed as the sum of two primes.

If-Then

"If John mows my lawn, I will pay him $20."

Truth Table for If-Then

A	B	If A, then B.
T	T	T
T	F	F
F	T	T
F	F	T

Other Names for "If A, then B."

- A implies B.
- $A \Rightarrow B$
- $B \Leftarrow A$
- Whenever A is true, B is true.
- A is sufficient for B.
- B is necessary for A.

If-Then

"If John mows my lawn, I will pay him $20."

Truth Table for If-Then

A	B	If A, then B.
T	T	T
T	F	F
F	T	T
F	F	T

Example

- The sum of two even integers is even.
- All trucks are vehicles.
- All vehicles are trucks.
- All nonvehicles are nontrucks.
- All differentiable functions are continuous.

Contrapositive and Converse

Assume the statement, “If A, then B,” is true.

- **Contrapositive:** “If not B, then not A.”
- The contrapositive is logically equivalent to the original statement, so it is also true.

- **Converse:** “If B, then A.”
- The converse is not logically equivalent to the original statement, so it may or may not be true.

If and Only If

- If x is an even integer, then $x + 1$ is an odd integer.
- If $x + 1$ is an odd integer, then x is an even integer.

- x is an even integer **if and only if** $x + 1$ is an odd integer.

"I will pay John $20, if and only if he mows my lawn."

Truth Table for If and Only If

A	B	A if and only if B.
T	T	T
T	F	F
F	T	F
F	F	T

If and Only If

Other Names for If and Only If

- A iff B.
- $A \Leftrightarrow B$
- A is necessary and sufficient for B.
- A is equivalent to B.

And

Truth Table for And

A	B	A and B.
T	T	T
T	F	F
F	T	F
F	F	F

Example

Which of these statements is true?

- $2+2=4$ and $\frac{d}{dx}\sin(x)=\cos(x)$
- $3|15$ and 1 is prime.
- 6 is an odd integer and 7 is a composite integer
- If x and y are integers such that $x^2+y^2=0$, then $x=0$ and $y=0$.

Or

Truth Table for Or

A	B	A or B.
T	T	T
T	F	T
F	T	T
F	F	F

Example

Which of these statements is true?

- $2+2=4$ or $\frac{d}{dx}\sin(x)=\cos(x)$
- $3|15$ or 1 is prime.
- 6 is an odd integer or 7 is a composite integer
- If x and y are integers such that $xy=0$, then $x=0$ or $y=0$.

Not

Truth Table for Not

A	Not A
T	F
F	T

Example

Which of these statements is true?

- It is not the case that $\frac{d}{dx}e^x = \cos(x)$
- 4 does not divide 20.

Contrapositives and Converses Revisited

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

- **Contrapositive:** (not B) $\Rightarrow$ (not A)
- **Converse:** $B \Rightarrow A$

Example

- Construct truth tables for the contrapositive and converse.
- This approach will help your with homework problems 4 and 6.

Vacuous Truth

Definition

- Consider the statement, “If A then B”.
- If it is impossible for A to be true, then the above statement is true.
- In this case, it is called *vacuously* true.

Example

- If an integer is both a perfect square and prime, then it is negative.
- If Santa Claus mows my lawn, I will pay him $1,000,000.
- All of my children are Nobel Prize winners.
- All of my children are convicted felons.

Homework

- **To Turn In:** p. 13 (1, 2, 4, 6, 7, 10, 12ace)
- **To Discuss:** p. 13 (1ab, 2ac, 4, 10, 12ce)

Outline

Our Starting Point: The Integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

You may use the following freely in any proof:

- Algebraic properties of addition, subtraction, and multiplication (not division).
- Basic number facts like $3 \times 2 = 6$.
- Basic facts about the order relations $(<, \leq, >, \geq)$.
- For specific details, see Appendix D.

Some Properties of the Integers

There exists a set $\mathbb{Z}$ called the **integers**, and binary operations defined on $\mathbb{Z}$ called **addition** and **multiplication** (denoted $x + y$ and xy), satisfying the following conditions:

For any integers x, y, and z,

- **Closure Property:** $x + y$ and xy are also integers
- **Commutative Properties:** $x + y = y + x$, and $xy = yx$
- **Associative Properties:** $x + (y + z) = (x + y) + z$ and $x(yz) = (xy)z$
- **Distributive Property:** $x(y + z) = xy + xz$

Additive Identity: There exists an integer 0, such that $x + 0 = x$, for any integer x.
Additive Inverse: For any integer x, there exists an integer $-x$, such that $x + (-x) = 0$.
Multiplicative Identity: There exists an integer 1, such that $1x = x$, for any integer x.

Some Properties of the Order Relations $<, >, \leq, \geq$

Let a, b, c, and d be integers.

- If $a < b$ and $c < d$, then $a + c < b + d$.
- Let x be a positive integer. Then $a < b$ if and only if $ax < bx$.
- **Transitive Property:** If $a < b$, and $b < c$, then $a < c$.
- The above properties are all true for $>$, $\leq$, and $\geq$ also.

Outline

Proposition 5.2

The sum of two even integers is even.

Proof Template 1: Direct Proof of an If Then statement.

To prove the statement, “If A, then B”

Assume A

$\vdots$

Make logical deductions

$\vdots$

Conclude B

Proposition 5.3

Let a, b, and c be integers. If $a|b$ and $b|c$, then $a|c$.

What can we say about $x^3 + 1$ if x is a positive integer? Prime or composite?

$$1^3 + 1 = 2$$
$$2^3 + 1 = 9$$
$$3^3 + 1 = 28$$
$$4^3 + 1 = 65$$
$$5^3 + 1 = 126$$

Proposition 5.4

Let x be an integer. If $x > 1$, then $x^3 + 1$ is composite.

Proposition 5.5

Let x be an integer. Then x is even if and only if $x + 1$ is odd.

Proof Template 2: Direct Proof of an If and Only If Statement.

To prove the statement, "*A* iff *B*"

- ($\Rightarrow$) Prove "If *A*, then *B*"
- ($\Leftarrow$) Prove "If *B*, then *A*"

Homework

- **To Turn In:** p. 22 (1–3, 5, 7–9, 15, 20, 24)
- **To Discuss:** p. 22 (1, 3, 7, 15, 20, 24)

Outline

False Statement

Let a and b be integers. If $a|b$ and $b|a$, then $a = b$.

Proof Template 3: Refuting a False If-Then Statement with a Counterexample

- To disprove a statement of the form “If A, then B”
- Find an instance where A is true but B is false.

Example

Disprove: If p and q are prime, then $p + q$ is composite.

Outline

And

$$\mathtt{TRUE} \wedge \mathtt{TRUE} = \mathtt{TRUE}$$
$$\mathtt{TRUE} \wedge \mathtt{FALSE} = \mathtt{FALSE}$$
$$\mathtt{FALSE} \wedge \mathtt{TRUE} = \mathtt{FALSE}$$
$$\mathtt{FALSE} \wedge \mathtt{FALSE} = \mathtt{FALSE}$$

Truth Table for $\wedge$

x	y	$x \wedge y$
TRUE	TRUE	TRUE
TRUE	FALSE	FALSE
FALSE	TRUE	FALSE
FALSE	FALSE	FALSE

Or

$$\text{TRUE} \vee \text{TRUE} = \text{TRUE}$$
$$\text{TRUE} \vee \text{FALSE} = \text{TRUE}$$
$$\text{FALSE} \vee \text{TRUE} = \text{TRUE}$$
$$\text{FALSE} \vee \text{FALSE} = \text{FALSE}$$

Truth Table for $\vee$

x	y	$x \vee y$
TRUE	TRUE	TRUE
TRUE	FALSE	TRUE
FALSE	TRUE	TRUE
FALSE	FALSE	FALSE

Not

$$\neg \texttt{TRUE} = \texttt{FALSE}$$
$$\neg \texttt{FALSE} = \texttt{TRUE}$$

Truth Table for ¬

x	$\neg x$
TRUE	FALSE
FALSE	TRUE

Example

Calculate the value of

$$\texttt{TRUE} \wedge ((\neg\texttt{FALSE}) \vee \texttt{FALSE})$$

Proposition 7.1: DeMorgan's Law

The Boolean expressions $\neg(x \wedge y)$ and $(\neg x) \vee (\neg y)$ are logically equivalent.

Proof Template 4: Truth Table of Logical Equivalence

To show that two Boolean expressions are logically equivalent:

- Construct a truth table showing the values of the two expressions for all possible values of the variables.
- Check to see that the two Boolean expressions always have the same value.

Theorem 7.2: Properties of Boolean Expressions

- $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$
- $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$
- $x \wedge \texttt{TRUE} = x$ and $x \vee \texttt{FALSE} = x$
- $\neg(\neg x) = x$
- $x \wedge x = x$ and $x \vee x = x$
- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
- $x \wedge (\neg x) = \texttt{FALSE}$ and $x \vee (\neg x) = \texttt{TRUE}$
- $\neg(x \wedge y) = (\neg x) \vee (\neg y)$ and $\neg(x \vee y) = (\neg x) \wedge (\neg y)$

Truth Table for $\rightarrow$

x	y	$x \rightarrow y$
TRUE	TRUE	TRUE
TRUE	FALSE	FALSE
FALSE	TRUE	TRUE
FALSE	FALSE	TRUE

Truth Table for $\leftrightarrow$

x	y	$x \leftrightarrow y$
TRUE	TRUE	TRUE
TRUE	FALSE	FALSE
FALSE	TRUE	FALSE
FALSE	FALSE	TRUE

Homework

- **To Turn In:** p. 24 (1–4, 6, 9b, 11) and p. 28 (1, 3, 11b, 13b)
- **To Discuss:** p. 24 (1, 3, 6, 9b) and p. 28 (1bc, 3, 11b)