Math 3320 Foundations of Mathematics

Chapter 2: Collections

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Outline

Definition

A *list* is an ordered sequence of objects.

- Order matters
- Repetitions are allowed

Definition

- The number of elements in a list is its *length*.
- A list of length two is called an *ordered pair*.
- A list of length n is called an n-tuple.
- The list of length zero is called the *empty list*, denoted $()$.
- Two lists are equal provided they have the same length and elements in corresponding positions in the two lists are equal.

Lists appear everywhere in mathematics!

Example

How many two-element lists can be made, where the first element is 1, 2, or 3, and the second element is 1, 2, 3, 4, or 5?

Example

How many two-element lists can be made, where both elements are 1, 2, 3, or 4, and repetitions don't occur?

Theorem 8.2: Multiplication Principle

Consider two-element lists for which

- there are n choices for the first element, and
- for each choice of the first element, there are m choices for the second element.

The the number of such lists is nm.

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Example

A person's initials are the two-element list consisting of the initial letters of their first and last names.

- How many possibilities are there for someone's initials?
- How many are there if repetitions aren't allowed?

Example

If a club has ten members, and they want to choose a president and vice president, how many outcomes are there?

Generalized Multiplication Principle

Consider a k-element lists for which

- there are n_i choices for the ith element, regardless of which choices were made for the previous elements.

Then the number of such lists is $n_1 n_2 \cdots n_k$.

Example

Joe has 5 shirts, 3 pairs of pants, and 2 pairs of shoes. If an outfit consists of choosing one shirt, one pair of pants, and one pair of shoes, how many outfits can Joe make?

Example

- How many three-element lists can be formed, if the elements are 1, 2, 3, 4, or 5?
- What if repetitions are not allowed?

Example

If a club has ten members, and they want to choose a president, vice president, secretary, and treasurer, how many outcomes are there?

Definition

Let $n \geq 0$ and $k > 0$ be integers. Then we define the *falling factorial* to be

$$(n)_k = n(n-1)(n-2)\cdots(n-k+1)$$

This is also known as the number of permutations of n objects, taken k at a time, denoted ${}_n\mathrm{P}_k$.

If $k = 0$, then we define $(n)_0 = 1$

Theorem 8.6

The number of lists of length k whose elements are chosen from a pool of n possible elements is

$$\begin{cases} n^k & \text{, if repetitions are permitted} \\ (n)_k & \text{, if repetitions are forbidden} \end{cases}$$

Example

- How many zero-element lists can be formed, if all elements in the list are 1, 2, 3, 4, or 5?
- What if repetitions are not allowed?

Example

- How many eight-element lists can be formed, if all elements in the list are 1, 2, 3, 4, or 5?
- What if repetitions are not allowed?

Outline

Definition

Let $n > 0$ be an integer. The *factorial* of n is

$$n! = (n)_n = n(n-1)(n-2)\cdots 1$$

If $n = 0$,

$$0! = (0)_0 = 1$$

$n!$ is the number of n-element lists whose elements are chosen from a poll of n objects.

Example

How many ways are there to rearrange the letters in the word “MATH”?

Product Notation

Let $x_1, x_2, \ldots, x_n$ be numbers. Then

$$\prod_{k=1}^{n} x_k = x_1 x_2 \cdots x_n.$$

Example

Calculate $\prod_{k=1}^{5}(2k+3)$

Write the following expressions using product notation.

- $n!$
- $(n)_k$
- n^k

Homework

- **To Turn In:** p. 38 (2, 3, 4, 5, 8, 9) and p. 42 (1, 2, 5, 6, 8)
- **To Discuss:** p. 38 (2, 5, 9) and p. 42 (1, 2)

Outline

Definition

- A *set* is a repetition-free, unordered collection of objects.
  - Order *doesn't* matter
  - Repetitions are *ignored*
- If x is an element of A, we write $x \in A$.
- If x is *not* an element of A, we write $x \notin A$.
- The number of elements in a set A is the *cardinality* of A, denoted $|A|$.
- A set is *finite* if its cardinality is an integer. Otherwise, it is called *infinite*.
- The *empty set* is the set with no elements, denoted $\{\}$, or $\emptyset$.

Set Builder Notation

$$\{\text{variable} : \text{conditions}\}$$

$$\{\text{variable} \mid \text{conditions}\}$$

Almost everything you study in mathematics is a set.

- **Linear Algebra**: Vector Spaces
- **Abstract Algebra**: Groups, Rings, and Fields
- **Topology**: Topological Spaces
- **Real Analysis**: Measure Spaces
- **Probability Theory**: Probability Spaces
- **Statistics:** Statistical Models
- **Differential Geometry**: Manifolds
- **Functional Analysis**: Banach Spaces, Hilbert Spaces
- **Differential Equations:** Sobolev Spaces
- Sets are **building blocks** used to build other mathematical objects:
  - Numbers
  - Functions
  - Vectors and Matrices
  - Ordered pairs and lists

Definition

- A set A is a *subset* of a set B if every element of A is also in B.
- Notation: $A \subseteq B$

Example

Define the set P of *Pythagorean Triples* as

$$P = \{(a, b, c) \mid a, b, c \in \mathbb{Z} \text{ and } a^2 + b^2 = c^2\}$$

- Is $(3, 4, 5) \in P$?
- Is $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1) \in P$?
- Define

  $$T = \{(p, q, r) \mid p = x^2 - y^2, q = 2xy, r = x^2 + y^2, \text{ where } x, y \in \mathbb{Z}\}$$

  Prove that $T \subseteq P$.

Proof Template 6: Proving one set is a subset of another

To show $A \subseteq B$

Assume $x \in A$

$\vdots$

Make logical deductions

$\vdots$

Conclude $x \in B$

Therefore, $A \subseteq B$

Example

Prove that $[1, 5) \subseteq [0, 8)$

Definition

If $A \subseteq B$, we also write $B \supseteq A$ (B is a *superset* of A).

Proposition

For any set A,

- $A \subseteq A$
- $\emptyset \subseteq A$

Proposition 10.3

Let x be any mathematical object and let A be a set. Then $x \in A$ iff $\{x\} \subseteq A$.

Definition

- Two sets are *equal* if they contain exactly the same elements.
- More formally, if A and B are sets, then $A = B$ means

$$x \in A \Leftrightarrow x \in B.$$

Proof Template 5: Proving two sets are equal

Let A and B be sets. To show $A = B$,

- Assume $x \in A$ and prove $x \in B$.
- Assume $x \in B$ and prove $x \in A$.

Example

Prove the following sets are equal:

- $E = \{x \in \mathbb{Z} \mid x \text{ is even}\}$
- $F = \{z \in \mathbb{Z} \mid z = a + b, \text{ where } a \text{ and } b \text{ are both odd}\}$

Proposition

Let A and B be sets. Then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Alternate Proof Template 5: Proving two sets are equal

Let A and B be sets. To show $A = B$,

- Prove $A \subseteq B$
- Prove $B \subseteq A$

Example

If $A = \{1, 2, 3\}$, how many subsets does A have?

Definition

- Let A be a set. The *power set* of A is the set of all subsets of A.
- Notation: 2^A or $\mathcal{P}(A)$.

Example

If $A = \{1, 2, 3\}$ what is 2^A?

Proposition

If A is a set, then

$$|2^A| = 2^{|A|}.$$

Homework

- **To Turn In:** p. 50 (1–4, 9, 10, 14)
- **To Discuss:** p. 50 (1af, 2ad, 3ag, 4abc, 9, 14)

Outline

Existential Quantifier

- $\exists x \in A$, assertion about x
- There exists $x \in A$, such that the assertions about x are true.
- Other phrases: there is, for some, for at least one

Example

Classify each as true/false.

- $\exists x \in \mathbb{Z}$, x is even and prime. True
- For some integer x, $x = -x$. True
- There is a positive integer x, such that $x^2 = -1$. False
- There exists $x \in \emptyset$, such that x is even and prime. False
- $\exists w \in \mathbb{Z}$, w is even and prime. True

$$\{x \in \mathbb{Z} \mid 1 \leq x \leq 4\} = \{q \in \mathbb{Z} \mid 1 \leq q \leq 4\}$$

Prove the following:

Proposition

There exists $x \in \mathbb{Z}$, such that x is even and prime.

Proof Template 7: Proving Existential Statements

To prove $\exists x \in A$, assertions about x:

- Choose a specific element $x \in A$.
- Prove that x satisfies the assertions.

Proposition

For some integer x, $x = -x$.

Universal Quantifier

- $\forall x \in A$, assertion about x
- For all $x \in A$, the assertions about x are true.
- Other phrases: for every, for each, for any, for an arbitrary

Example

Classify each as true/false.

- $\forall x \in \mathbb{Z}$, x is even and prime. False
- For all integers x, $x + 0 = x$. True
- For any $a, b \in \mathbb{Z}$, $(a + b)^2 = a^2 + b^2$. False
- For each $x \in \emptyset$, $x^2 = -5$. True (Vacuously)
- For arbitrary $a, b, c \in \mathbb{Z}$, $a(b + c) = ab + ac$. True
- $\forall x \in \mathbb{Z}, \sqrt{x^2} = x$. False
- $\forall x > 0, \sqrt{x^2} = x$. True

Prove the following:

Proposition

Let $A = \{x \in \mathbb{Z} \mid 6|x\}$. Then $\forall x \in A$, x is even.

Proof Template 8: Proving Universal Statements

To prove $\forall x \in A$, assertions about x:

- Let $x \in A$.
- Prove that x satisfies the assertions.

Proposition

For every $x \in \mathbb{Z}$, $x|x$.

Negations

- Let P be a mathematical statement.
- The *negation* of P is $\neg P$.
- $\neg P$ is true iff P is false.

Example

- The statement $2 + 2 = 4$ is True.
- Its negation is $\neg(2 + 2 = 4)$ is False.

Negations of Quantified Statements

- $\neg(\exists x \in A, \text{assertions about } x) = \forall x \in A, \neg(\text{assertions about } x)$
- $\neg(\forall x \in A, \text{assertions about } x) = \exists x \in A, \neg(\text{assertions about } x)$
- $\neg\exists \ldots = \forall\neg \ldots$
- $\neg\forall \ldots = \exists\neg \ldots$

Negations of Quantified Statements

- $\neg\exists \ldots = \forall\neg \ldots$
- $\neg\forall \ldots = \exists\neg \ldots$

Example

Prove that the statement $\forall a, b \in \mathbb{Z}, (a+b)^2 = a^2 + b^2$ is false.

Proof Template: Proving that a statement is false

To prove that a statement is false, prove that its negation is true.

Example

Prove that the statement $\exists x \in \mathbb{Z}, x^2 = -1$ is false.

Proof Template 3: Refuting a False If-Then Statement with a Counterexample

- To disprove a statement of the form "If *A*, then *B*". $\forall x, A \Rightarrow B$
- Find an instance where *A* is true but *B* is false. $\exists x, A \wedge (\neg B)$

Proof Template: Proving that a statement is false

To prove that a statement is false, prove that its negation is true.

Combining Quantifiers

Example

Classify each as true/false. Assume all variables represent integers.

- $\forall x, \exists y, x + y = 0$
- $\exists x, \forall y, x + y = 0$
- Prove the first statement is true and the second statement is false.
- $\forall x, \forall y, x + y = 0$
- $\exists x, \exists y, x + y = 0$

Homework

- **To Turn In:** p. 54 (1a–d, 2a–d, 4e–h, 5a–d, 7a–d)
- **To Discuss:** p. 54 (1ab, 2ab, 4fg, 5ab, 7ab)

Outline

Definition

- Let A and B be sets.
- The *union* of A and B is

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

- The *intersection* of A and B is

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Theorem 12.3

Let A, B, and C be sets. Then,

- $A \cup B = B \cup A$ and $A \cap B = B \cap A$
- $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$
- $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Theorem 7.2: Properties of Boolean Expressions

- $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$
- $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$
- $x \wedge \mathtt{TRUE} = x$ and $x \vee \mathtt{FALSE} = x$
- $\neg(\neg x) = x$
- $x \wedge x = x$ and $x \vee x = x$
- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
- $x \wedge (\neg x) = \mathtt{FALSE}$ and $x \vee (\neg x) = \mathtt{TRUE}$
- $\neg(x \wedge y) = (\neg x) \vee (\neg y)$ and $\neg(x \vee y) = (\neg x) \wedge (\neg y)$

Definition

- The sets A and B are *disjoint* provided $A \cap B = \emptyset$.
- The sets $A_1, A_2, \ldots, A_n$ are *pairwise disjoint* or *mutually exclusive* if, for all $i, j = 1, \ldots, n$,

$$i \neq j \Rightarrow A_i \cap A_j = \emptyset.$$

Theorem: Addition Principle

- If A and B are disjoint, then

$$|A \cup B| = |A| + |B|$$

- If $A_1, A_2, \ldots, A_n$ are *pairwise disjoint*, then

$$\left| \bigcup_{k=1}^{n} A_k \right| = \sum_{k=1}^{n} |A_k|.$$

Definition

- Let A and B be sets.
- The *set difference* is

$$A - B = \{x \in A \mid x \notin B\}$$

- The *symmetric difference* is

$$A \Delta B = (A - B) \cup (B - A).$$

Proposition

If A and B are sets, then $A \cap B$ and $A - B$ are disjoint, and

$$(A \cap B) \cup (A - B) = A.$$

Proposition 12.4

Let A and B be finite sets. Then,

$$|A| + |B| = |A \cup B| + |A \cap B|.$$

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Example

How many integers between 1 and 1000 are divisible by 2 or 5?

Proposition 12.12: DeMorgan's Laws

If A, B, and C are sets, then

- $A - (B \cup C) = (A - B) \cap (A - C)$
- $A - (B \cap C) = (A - B) \cup (A - C)$

Definition

If A and B are sets, then their *Cartesian product* is

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Example

If $A = \{1, 2, 3\}$, and $B = \{7, 8\}$, what are $A \times B$ and $B \times A$?

Homework

- **To Turn In:** p. 64 (1, 2, 5(3rd part only), 6, 11, 12, 21af, 22)
- **To Discuss:** p. 64 (1ade, 5(3rd part only), 6, 11, 22)