Math 3320 Foundations of Mathematics

Chapter 3: Counting and Relations

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Outline

Definition

- A *relation* R is a set of ordered pairs.
- Let A and B be sets. We say R is a relation *from A to B* if

$$R \subseteq A \times B.$$

- We say R is a relation on A if

$$R \subseteq A \times A.$$

Definition

Let R be a relation. The *inverse* of R, denoted R^{-1}, is the relation formed by reversing the order of all the ordered pairs in R.

$$R^{-1} = \{(x, y) \mid (y, x) \in R\}.$$

Definition (Properties of Relations)

Let R be a relation on a set A.

- If $\forall x \in A : xRx$, we call R *reflexive*.
- If $\forall x \in A : x \not R x$, we call R *irreflexive*.
- If $\forall x, y \in A : xRy \Rightarrow yRx$, we call R *symmetric*.
- If $\forall x, y \in A : (xRy \wedge yRx) \Rightarrow x = y$, we call R *antisymmetric*.
- If $\forall x, y, z \in A : (xRy \wedge yRz) \Rightarrow xRz$, we call R *transitive*.

Example

- Which of the above properties does the relation $=$ satisfy?
- Which of the above properties does the relation $\leq$ satisfy?

Homework

- **To Turn In:** p. 76 (1ab, 2a, 4, 6, 7ac, 9, 10, 14, 15)
- **To Discuss:** p. 76 (1a, 4, 6abf, 7a, 9, 10, 14, 15)

Outline

Definition (Equivalence Relation)

Let R be a relation. We say R is an *equivalence relation* provided it is

- reflexive,
- symmetric,
- transitive.

Definition

Let n be a positive integer. We say that *integers* x and y are *congruent modulo* n, and we write

$$x \equiv y \pmod{n}$$

if $n|(x-y)$.

Proposition

- Congruence (mod n) is an equivalence relation.
- $x \equiv 0 \pmod{2}$ iff x is even
- $x \equiv 1 \pmod{2}$ iff x is odd

Definition (Equivalence Classes)

Let R be an equivalence relation on a set A, and let $a \in A$. The *equivalence class of a*, denoted $[a]$, is

$$[a] = \{x \in A \mid xRa\}.$$

Proposition

Let $[a]$ be an equivalence class.

- $a \in [a]$
- If $b \in [a]$, then $[a] = [b]$.

Proposition

Let R be an equivalence relation on a set A. The equivalence classes of R are nonempty, pairwise disjoint subsets of A whose union is A.

Homework

- **To Turn In:** p. 83 (1ab, 3, 7, 8abd, 11)
- **To Discuss:** p. 83 (1a, 3acg, 7ad, 8ab, 11)

Outline

Definition

Let A be a set. A *partition of A* is a set of nonempty, pairwise disjoint sets whose union is A.

Example

Let $A = \{1, 2, 3, 4, 5, 6\}$. Which of the following are partitions?

- $\{\{1, 2\}, \{3\}, \{4, 5, 6\}\}$
- $\{\{1, 2, 3, 4\}, \{3, 4, 5, 6\}\}$
- $\{\{1, 2, 3\}, \{4\}, \{5\}\}$
- $\{\{\}, \{1, 4\}, \{2, 3, 5, 6\}\}$
- $\{\{1, 2, 3, 4, 5, 6\}\}$
- $A = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$

Proposition

Let R be an equivalence relation on a set A. The equivalence classes of R form a partition of the set A.

- Let A be a set and $\mathcal{P}$ be a partition on A.
- Define
$$a \overset{\mathcal{P}}{\equiv} b \text{ if and only if } \exists P \in \mathcal{P}, a, b \in P.$$
- $\overset{\mathcal{P}}{\equiv}$ is an equivalence relation on A.
- The equivalence classes of $\overset{\mathcal{P}}{\equiv}$ are the elements of $\mathcal{P}$.

Homework

- **To Turn In:** p. 89 (1)
- **To Discuss:** p. 89 (1)