Math 3320 Foundations of Mathematics

Chapter 4: More Proof

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Outline

Proof by Contrapositive

Setting:

- We want to prove, "If A, then B".
- This is logically equivalent to, "If $\neg B$, then $\neg A$".

Proof template

Assume $\neg B$

$\vdots$

Deduce $\neg A$

Proposition

Let R be an equivalence relation on a set A, and let $a, b \in A$.

If $a \not R b$, then $[a] \cap [b] = \emptyset$.

Proof by Contradiction

Setting: We want to prove A.

Proof template

Assume $\neg A$

$\vdots$

Show that this leads to a contradiction.

Proof by Contradiction for If-then Statements

Setting: We want to prove $A \Rightarrow B$.

Proof template

Assume A and $\neg B$

$\vdots$

Show that this leads to a contradiction.

Proving that a Set is Empty

Setting: We want to prove the set A is empty.

Proof template

Assume $A \neq \emptyset$

$\vdots$

Show that this leads to a contradiction.

Proving Uniqueness

Setting: We want to prove there is at most one object that satisfies certain conditions.

Proof template

Assume x and y are objects satisfying the conditions.

$\vdots$

Deduce that $x = y$.

Homework

- **To Turn In:** p. 124 (1abc, 2, 3, 4abc, 7, 13)
- **To Discuss:** p. 124 (1a, 2, 3, 4a, 7, 13)

Outline

Proposition

Let n be a natural number. Then,

$$0^2 + 1^2 + 2^2 + \cdots + n^2 = \frac{(2n+1)(n+1)(n)}{6}.$$

Principle of Mathematical Induction

Let $A \subseteq \mathbb{N}$. If

- $0 \in A$, and
- $\forall k \in \mathbb{N}, k \in A \Rightarrow k+1 \in A$,

then $A = \mathbb{N}$.

Strong Induction

Let $A \subseteq \mathbb{N}$. If

- $0 \in A$, and
- $\forall k \in \mathbb{N} : 0, 1, 2, \ldots, k \in A \Rightarrow k + 1 \in A$,

then $A = \mathbb{N}$.

Homework

- **To Turn In:** p. 146 (4abcdeh, 5aef)
- **To Discuss:** p. 146 (4abeh, 5aef)

Outline

Definition

- A *relation* R is a set of ordered pairs.
- Let A and B be sets. We say R is a relation *from A to B* if

$$R \subseteq A \times B.$$

Definition

- Let f be a relation from A to B.
- We say that f is a *function* if

$$\forall a \in A, \exists! b \in B, (a, b) \in f.$$

- In this case, we write $f : A \to B$.
- Instead of $(a, b) \in f$, we usually write $f(a) = b$.

Definition

- Let $f : A \to B$.
- A is called the *domain* of f, denoted $\operatorname{dom} f$.
- B is called the *codomain* of f.
- The *image* or *range* of f is

$$\operatorname{im} f = \{b \mid \exists a \in A, f(a) = b\}$$

Definition

- We say that f is *one-to-one* or *injective*, if

$$\forall a_1, a_2 \in A : f(a_1) = f(a_2) \Rightarrow a_1 = a_2.$$

- We say that f is *onto* or *surjective*, if

$$\forall b \in B : \exists a \in A : f(a) = b.$$

- If f is injective and surjective, we say it is *bijective*.

Definition

- Let A be a set.
- If there is a bijection $f : \mathbb{N} \to A$, we say that A is *countably infinite*.
- If A is finite or countably infinite, we say it is *countable*. Otherwise, A is *uncountably infinite*.

Homework

- **To Turn In:** p. 175 (1a–g Omit part (4), 5, 6, 14)
- **To Discuss:** p. 175 (1beg Omit part (4), 5ab, 6ab, 14ab)