

Math 3320 Foundations of Mathematics

Real Analysis: Limits and Continuity

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Example

What is $\lim_{x \rightarrow 5} 7x$?

What does it mean to say that

$$\lim_{x \rightarrow a} f(x) = L?$$

Limits as a Game

- Player 1: Chooses an interval containing L .
- Player 2: Tries to choose an interval containing a that the function f maps into the first interval. **We don't allow $x = a$**
- If it's always possible for Player 2 to succeed, then $\lim_{x \rightarrow a} f(x) = L$.
- If Player 1 can create an interval where Player 2 can't succeed, then $\lim_{x \rightarrow a} f(x) \neq L$.

Definition

- Let f be a function defined on some interval containing a , except possibly at a itself.
- Then $\lim_{x \rightarrow a} f(x) = L$ means that

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}, 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

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Useful Facts About Absolute Value

For any $x, y \in \mathbb{R}$

- $|xy| = |x||y|$
- $|x + y| \leq |x| + |y|$

- $\lim_{x \rightarrow 5} 7x = 7(5) = 35$
- $\lim_{x \rightarrow a} x^2 = a^2$
- $\lim_{x \rightarrow a} \sin(x) = \sin(a)$
- $\lim_{x \rightarrow a} e^x = e^a$

Definition

- A function is *continuous* at a , if

$$\lim_{x \rightarrow a} f(x) = f(a).$$

- Otherwise, it is *discontinuous* at a .

Answer each of the following, and prove that your answer is correct.

- 1 Is $f(x) = 5x + 8$ continuous at 3?
- 2 Let $a, c \in \mathbb{R}$, and define $f(x) = cx^3$. Is f continuous at a ?
- 3 Define the function f as follows:

$$f(x) = \begin{cases} \sqrt{x} & , \text{ if } x \geq 25 \\ 2 & , \text{ if } x < 25. \end{cases}$$

Plot f . Is f continuous at 25?