Trig Equation with Inverse Trig Function

In this example we solve the equation

$$\arcsin x - \arccos x = \frac{\pi}{6}$$

Before beginning the solution steps, we observe two facts about x:

1. The domain of the arcsin and arccos function is $[-1, 1]$, and so any solution x of this equation must be between -1 and 1, that is,

$$-1 \le x \le 1$$

2. If x is a negative number, then $\arcsin(x)$ is in the interval $\left[-\frac{\pi}{2}, 0\right)$ and $\arccos(x)$ is in the interval $\left(\frac{\pi}{2}, \pi\right]$. This leads to the observation that if $x < 0$,

$$\arcsin x - \arccos x < 0 < \frac{\pi}{6}$$

because we are subtracting a positive number from a negative number. Any solution x of this equation must be in the interval $[0, 1]$:

$$0 \le x \le 1$$

With the observation that a solution must be a positive number ($x = 0$ is not a solution), we begin the solution steps

$$\arcsin x - \arccos x = \frac{\pi}{6}$$

$$\arcsin x = \arccos x + \frac{\pi}{6} \qquad \text{add } \arccos x \text{ to both sides}$$

$$\sin(\arcsin x) = \sin\left(\arccos x + \frac{\pi}{6}\right) \qquad \text{take the sine of both sides of the equation}$$

$$x = \sin(\arccos x)\cos\left(\frac{\pi}{6}\right) + \cos(\arccos x)\sin\left(\frac{\pi}{6}\right) \qquad \text{use } \sin(A+B) \text{ formula}$$

$$x = \sqrt{1-x^2}\left(\frac{\sqrt{3}}{2}\right) + x \cdot \frac{1}{2} \qquad \text{remember that } \arccos x \text{ is in QI}$$

$$2x = \sqrt{1-x^2}\,\sqrt{3} + x \qquad \text{multiply both sides by 2}$$

$$x = \sqrt{1-x^2}\,\sqrt{3} \qquad \text{subtract } x$$

$$x^2 = (1-x^2)(3) \qquad \text{square both sides of the equation}$$

$$x^2 = 3 - 3x^2 \qquad \text{distributive law}$$

$$4x^2 = 3 \qquad \text{add } 3x^2 \text{ to both sides}$$

$$x^2 = \frac{3}{4} \qquad \text{divide both sides by 4}$$

$$x = \frac{\sqrt{3}}{2} \text{ or } x = -\frac{\sqrt{3}}{2} \qquad \text{take square root of both sides}$$

Because the solution is a positive number, it must be the case that

$$x = \frac{\sqrt{3}}{2}$$

is the only **possible** solution. Since we squared both sides of the equation in one of the steps, we must check this answer to insure that it really balances the original equation.

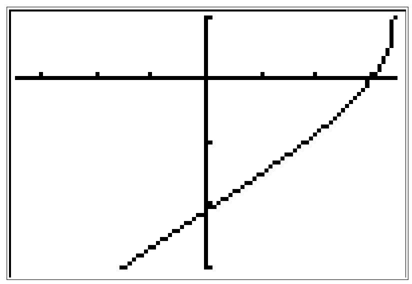
$$\arcsin\frac{\sqrt{3}}{2} - \arccos\frac{\sqrt{3}}{2} = \frac{\pi}{3} - \frac{\pi}{6}$$
$$= \frac{\pi}{6}$$

This confirms that the solution set is

$$\left\{\frac{\sqrt{3}}{2}\right\}$$

Here's a screenshot with $Y_1 = \sin^{-1}(X) - \cos^{-1}(X) - \pi/6$

The Window used here is
Xmin = -1, Xmax = 1, Xscl = √(3)/6, Ymin = -3, Ymax = 1, Yscl = 1



The only x-intercept is at 3√(3)/6 = $\frac{\sqrt{3}}{2}$ and gives visual support to the solution found analytically.