Exponential Functions

An **exponential function** $Q = f(t)$ has the formula

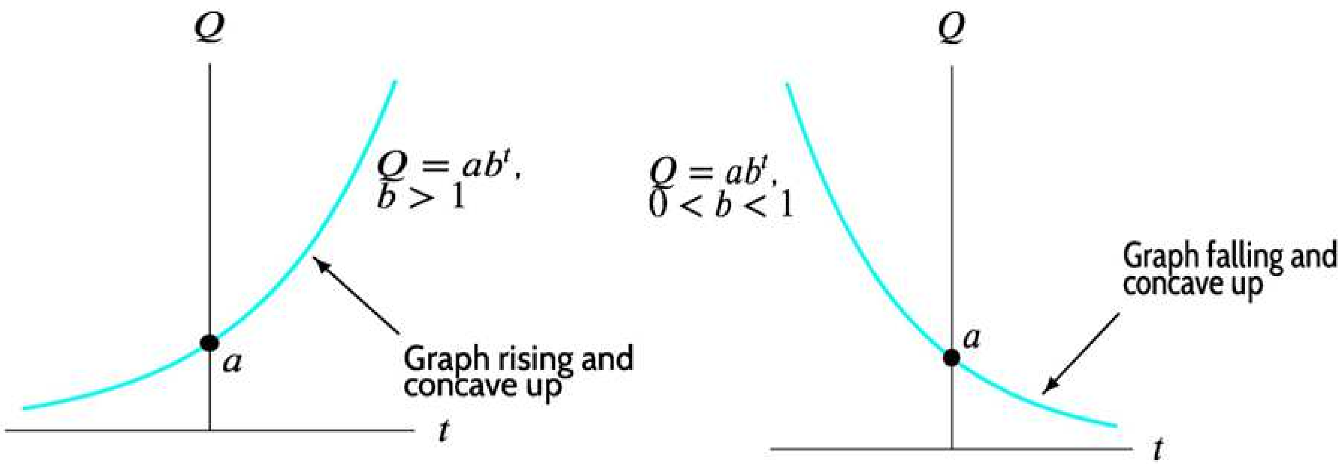
$$f(t) = a \cdot b^t, a \neq 0, b > 0$$

where a is the *initial value* of Q (at $t = 0$) and b, the base, is the *growth factor*. The growth factor is given by

$$b = 1 + r$$

where r is the decimal representation of the percent rate of change. For $a > 0$,
• if there is exponential growth, then $r > 0$ and $b > 1$.
• if there is exponential decay, then $r < 0$ and $0 < b < 1$.

There are two basic forms for the graph of an exponential function:



Example: Determine whether a function given by a table is linear or exponential. Find a possible formula in each case.

x	0	1	2	3	4
$f(x)$	20	16	12	8	4
$g(x)$	12	9.6	7.68	6.144	4.9152

Solution: The *difference* of consecutive y-values is constant for function f and so the table could represent a linear function in this case. We find the slope between successive points:

$$m = \frac{16 - 20}{1 - 0} = -4$$

Since we know the y-intercept is 20, we can write

$$f(x) = -4x + 20$$

and check this against the table entries.

In the case of function g, we examine the *ratio* of consecutive y-values:

$$9.6/12,\ 7.68/9.6,\ 6.144/7.68,\ \text{etc.}$$

In all cases we get a constant common ratio of 0.8, so that each y-value changes by a factor of 0.8. This indicates an exponential function.

To find a formula for this exponential function, we note that the base $b = 0.8$ and the initial value at $x = 0$ is 12. The formula

$$g(x) = 12 \cdot 0.8^x$$

will produce the given table of values for $g(x)$.

Example: Find a formula for the exponential function that satisfies $f(4) = 16$ and $f(7) = 45.2548$.

Solution: The function $f(x) = a \cdot b^x$, and we must find the values of a and b. First write two equations

$$ab^7 = 45.2548$$
$$ab^4 = 16$$

Dividing, we can write

$$\frac{ab^7}{ab^4} = \frac{45.2548}{16}$$

and so

$$b^3 = 2.828425$$
$$b = 2.828425^{1/3} = 1.4142$$

We complete the problem by solving for a, and writing the formula for f

$$a \cdot 1.4142^4 = 16$$
$$a = 4.0001$$
$$f(x) = 4.0001 \cdot 1.4142^x$$

Example: Find the initial value a, growth/decay factor b, andgrowth/decay rate r for the exponential function

$$Q(t) = 0.057(2.5)^{-4t}$$

Solution: Use the exponent laws to write

$$\begin{aligned} Q(t) &= 0.057(2.5)^{-4t} \\ &= 0.057\left((2.5)^{-4}\right)^t \\ &= 0.057(0.0256)^t \end{aligned}$$

From this form we have

$$a = 0.057$$
$$b = 0.0256$$
$$r = b - 1 = -0.9744$$

The number r is commonly written as a *percent*:

$$r = -97.44\%$$

Another way to express r is to say "the *decay rate* is 97.44%."

Example: Twenty-five percent of a radioactive substance decays in ten years. By *what percent* does the substance decay each year?

Solution: We don't have a numeric initial amount given, and so we will call it a. We know that after 10 years, 75 percent of the substance remains:

$$0.75a = ab^{10}$$

We can divide both sides of the equation by a and write

$$0.75 = b^{10}$$
$$b = 0.75^{1/10}$$
$$b = 0.9716$$

Since $r = b - 1$, we get

$$r = 0.9716 - 1 = -0.0284$$

We can say that

$$r = -2.84\%$$

or the **decay rate** is 2.84% per year.