Notes: Law of Cosines

There are two triangle situations in which the Law of Sines will not give a solution. The first is **SAS**–two sides and the angle **between** them is given. In this situation we do not have an opposite side for the angle and so we cannot construction a proportion equation. The second is **SSS**–three sides but no angles given. Again, we don't have an opposite angle that is needed to use the Law of Sines. In these situations we apply the **Law of Cosine**s. We consider the SAS case first.

SAS

Law of Cosines: If sides *a* and *b* are given, and the angle *C* between them, then

$$c^2 = a^2 + b^2 - 2ab\cos C$$

"A side squared equals the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the angle between them."

Does it look like the Pythagorean Theorem? Indeed, if $\angle C = 90°$ then $\cos C = 0$ and in this special case

$$c^2 = a^2 + b^2$$

Example: Suppose you are given a triangle with

$$A = 60°, b = 6, c = 9$$

Find side *a* and angles *B* and *C*.

Solution: We set up the Law of Cosines with the pattern above

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$a^2 = 6^2 + 9^2 - 2(6)(9)\cos 60°$$
$$a^2 = 63$$
$$a = \sqrt{63}$$
$$a = 7.9373$$

With the calculator, we will store this result in [A] and use it to find the other angle.

Procedure: After finding the side, you should find the *smaller* of the remaining two angles. It will be the angle opposite the smaller of the two sides. The smaller angle must be in QI, and so the inverse sine will give us the answer when we use the Law of Sines.

In our case, that's angle *B*.

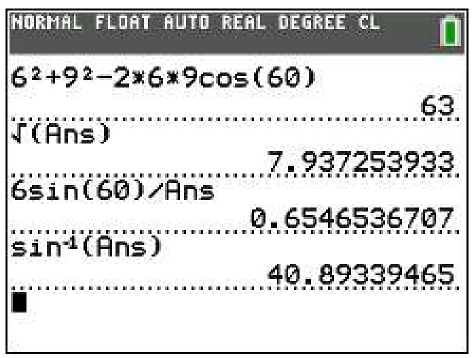
"I didn't start with the Law of Sines, but I got there as quick as I could."

$$
\begin{aligned}
\frac{a}{\sin A} &= \frac{b}{\sin B} \\
\sin B &= \frac{b \sin A}{a} \\
&= \frac{6 \sin 60^\circ}{7.9373} \\
&= 0.65465 \\
B &= \sin^{-1}(0.65465) \\
&= 40.893^\circ
\end{aligned}
$$

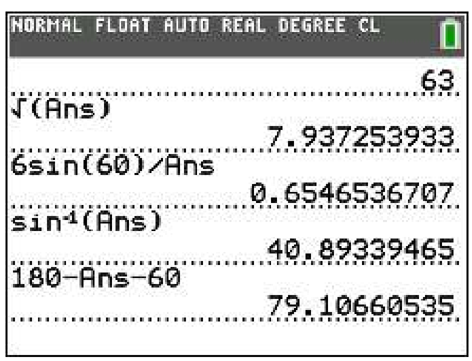
To finish the problem find angle C. No Law of Sines needed.

$$
\begin{aligned}
C &= 180^\circ - A - B \\
&= 180^\circ - 60^\circ - 40.893^\circ \\
&= 79.107^\circ
\end{aligned}
$$

Here's a calculator view. Note the use of the **ANS** variable. It must be used immediately in each line so that it will have the correct value.



Finish this by subtracting the angles from 180 degrees.



SSS

When we are given three sides we are not guaranteed that there is a triangle with those measurements. To create a triangle:

The largest side must be **less** than the sum of the two other sides.

Example: Which set of measures can be the lengths of sides of a triangle?
A. 2 cm, 2 cm, 5 cm
B. 1 cm, 2 cm, 3 cm
C. 4 cm, 6 cm, 12 cm
D. 3 cm, 5 cm, 7 cm

Solution: Only answer (C) meets the requirements.

To prepare for the SSS example, we solve the Law of Cosines for $\cos C$ to get

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

When we get the lengths of the sides, we want to use the Law of Cosines to find the **largest** angle first. This angle may be obtuse, in QII. The inverse cosine function will give an answer in QII without needing reference angles. Then we can get back to the Law of Sines to find the smallest angle. We then use these to find the third angle.

Example: Consider the triangle which has

$$a = 5, b = 6 \text{ and } c = 9$$

Find the measure of angles $\angle A$, $\angle B$ and $\angle C$. Give your answer in degrees to at least 3 decimal places.

Solution: First we find angle C, the largest angle (opposite the largest side).

$$\begin{aligned} \cos C &= \frac{a^2 + b^2 - c^2}{2ab} \\ &= \frac{5^2 + 6^2 - 9^2}{2(5)(6)} \\ &= -0.33333 \\ C &= \cos^{-1}(-0.33333) \\ &= 109.471^\circ \end{aligned}$$

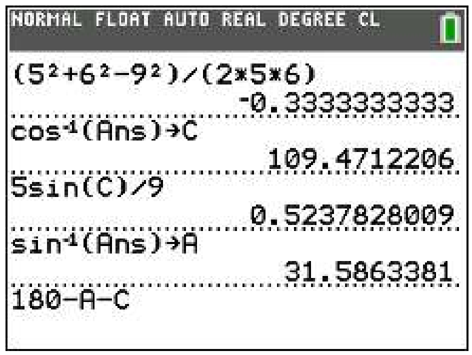
The inverse cosine of a negative number is a QII (obtuse) angle. Now we can use the Law of Sines to find the smallest angle A.

$$\begin{aligned} \frac{c}{\cos C} &= \frac{a}{\sin A} \\ \sin A &= \frac{a \sin C}{c} \\ &= \frac{5 \sin 109.471^\circ}{9} \\ &= 0.52379 \\ A &= \sin^{-1}(0.52379) \\ &= 31.586^\circ \end{aligned}$$

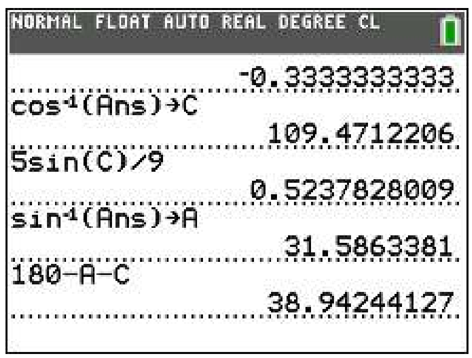
Then find the third angle, angle B

$$\begin{aligned} B &= 180° - A - C \\ &= 180° - 31.586° - 109.471° \\ &= 38.942° \end{aligned}$$

Here's the calculator view. Again, use the **ANS** variable carefully from step to step and [STO->] the result using the [ALPHA] key.



This last line, when entered, will give angle B, and scroll the top line off the screen.



SSS Application: Area with Heron's Formula

If sides a, b, and c are given, then the area of the triangle is given by

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)} \text{ where}$$

$$s = \frac{a+b+c}{2}$$

The number s is called the **semi-perimeter**.

The proof of this is given in the Math 1316 textbook and takes two pages!

Example: A triangular parcel of land has sides of lengths 370 feet, 650 feet and 640 feet. What is its area?

Solution: First find the semi-perimeter

$$s = \frac{370 + 650 + 640}{2}$$

$$s = 830$$

Then find the area.

$$\text{Area} = \sqrt{830(830 - 370)(830 - 650)(830 - 640)}$$

$$= 114269.7\,\text{ft}^2$$

If land is valued at \$1600 per acre [must be desert] (1 acre is 43,560 ft^2), what is the value of the parcel of land?

Multiply the area in ft^2 by $\frac{1\,\text{acre}}{43560\,\text{ft}^2}$ and then by $\frac{\$1600}{\text{acre}}$ to get the value in dollars:

$$114269.7\,\text{ft}^2 \cdot \frac{1\,\text{acre}}{43560\,\text{ft}^2} \cdot \frac{\$1600}{\text{acre}} = \$4197.23$$

Here's a calculator screenshot to show the use of the **ANS** variable.