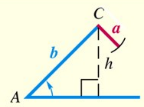
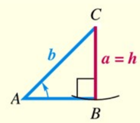
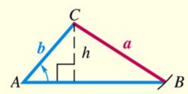
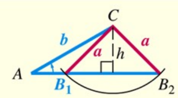
Notes: Law of Sines Ambiguous Case

In the previous note set we considered problems in which two angles and a side were given (ASA). Next we turn our attention to the SSA case where two sides and an angle opposite one of the sides is given. In the illustrations that follow we use sides a and b as given, and angle A as the opposite angle.

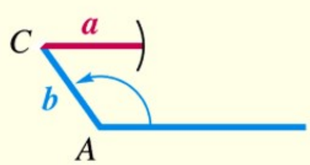
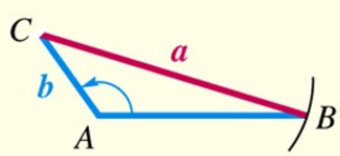
Case I. Angle A is acute

In these pictures the height $h = b\sin A$. Then compare a, b, and h. There are four possible situations:

Number of Triangles	Sketch	Applying Law of Sines Leads to
0		$\sin B > 1$, $a < h < b$
1		$\sin B = 1$, $a = h$ and $h < b$
1		$0 < \sin B < 1$, $a \geq b$
2		$0 < \sin B_2 < 1$, $h < a < b$

Case II Angle A is obtuse

In this situation we do not need to find the height h. We only need to compare a and b. There are two possible situations.

Number of Triangles	Sketch	Applying Law of Sines Leads to
0		$\sin B \geq 1$, $a \leq b$
1		$0 < \sin B < 1$, $a > b$

[These two illustrations from the Math 1316 textbook Trigonometry, 11th Edition, authors Margaret Lial, John Hornsby, David I. Schneider, Callie Daniels. Published by Pearson]

Here's a link to a GeoGebra applet that can be used to illustrate the cases for an acute angle A.

https://www.geogebra.org/m/FMRn65yc [copy and paste this link into your browser]

Example: For each of the following combinations of A, a, and b indicate if there are 0, 1, or 2 triangles with these data.

1. $A = 30°, b = 10, a = 12$: number = ?
2. $A = 30°, b = 10, a = 8$: number = ?
3. $A = 30°, b = 10, a = 3$: number = ?

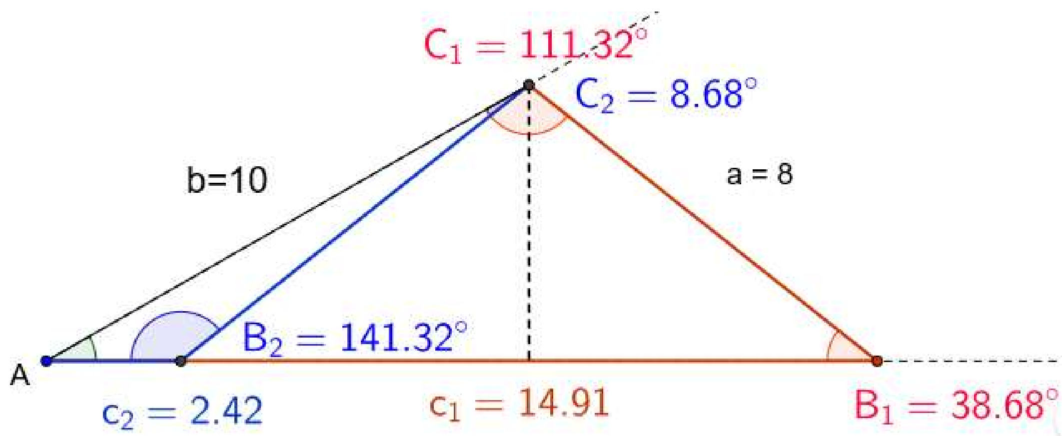
Solution: In all three cases we have

$$h = 10 \sin 30°$$
$$h = 5$$

With $h = 5$ and $b = 10$ we have:

1. The side $a = 12$, and so $a \geq b$. There is one triangle possible.
2. The side $a = 8$, and so $h < a < b$. There are two triangles possible.

Here's a picture from the GeoGebra applet.



3. The side $a = 3$, and so $a < h < b$. No triangle is possible.

Example: There are two triangles for which

$$A = 30°, a = 7, \text{ and } b = 10$$

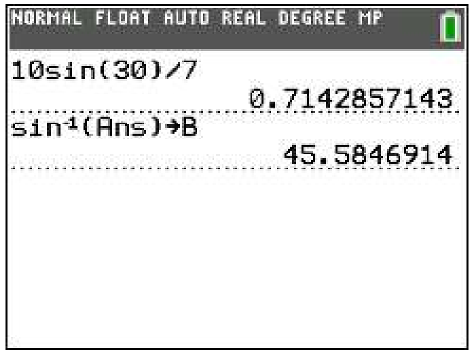
Completely solve both triangles.

Solution: Here's the steps for the first triangle where B is an acute angle.

Set up the Law of Sines to find $\sin B$:

$$\frac{7}{\sin 30°} = \frac{10}{\sin B}$$
$$\sin B = \frac{10 \sin 30°}{7}$$
$$\sin B = 0.71429$$
$$B = \sin^{-1}(0.71429)$$
$$B = 45.585°$$

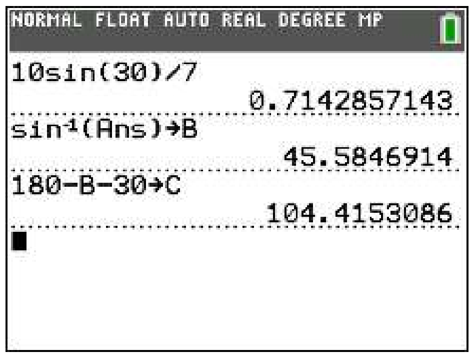
We need to store this value for later use. Be sure your calculator is in **DEGREE** mode.



Use [STO->] and [ALPHA][B] Now find angle C:

$$C = 180° - B - 30° = 104.415°$$

and store it in [ALPHA][C] on the calculator.



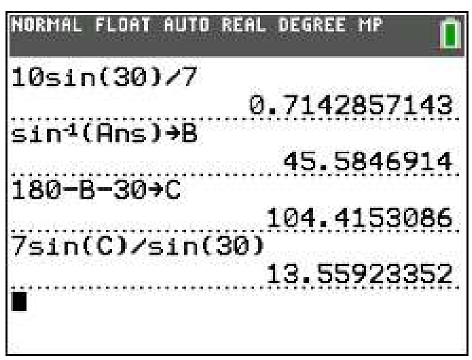
We can now use the Law of Sines to find side c.

$$\frac{7}{\sin 30°} = \frac{c}{\sin 104.415°}$$

$$c = \frac{7 \sin 104.415°}{\sin 30°}$$

$$c = 13.559$$

On the calculator use [C] instead of typing the numeric angle.



This completes the solution of the first triangle with

$$B = 45.584°$$
$$C = 104.415°$$
$$c = 13.559$$

Now it's time to find the **second** triangle. We will need the value of angle B we found earlier. The equation

$$\sin B = 0.71429$$

has two solutions. The first is in QI

$$B = 45.585°$$

The second in QII has this value of B as a **reference angle** so that the second solution, called B', is

$$B' = 180° - B$$
$$B' = 180° - 45.585°$$
$$B' = 134.42°$$

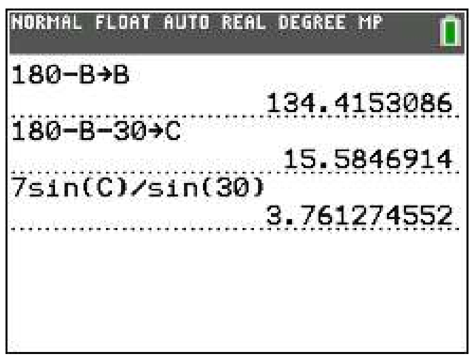
Then we have C'

$$C' = 180° - B' - 30° = 15.585°$$

and then side c' following the same steps as before.

$$c' = \frac{7\sin 15.585°}{\sin 30°}$$
$$c' = 3.7613$$

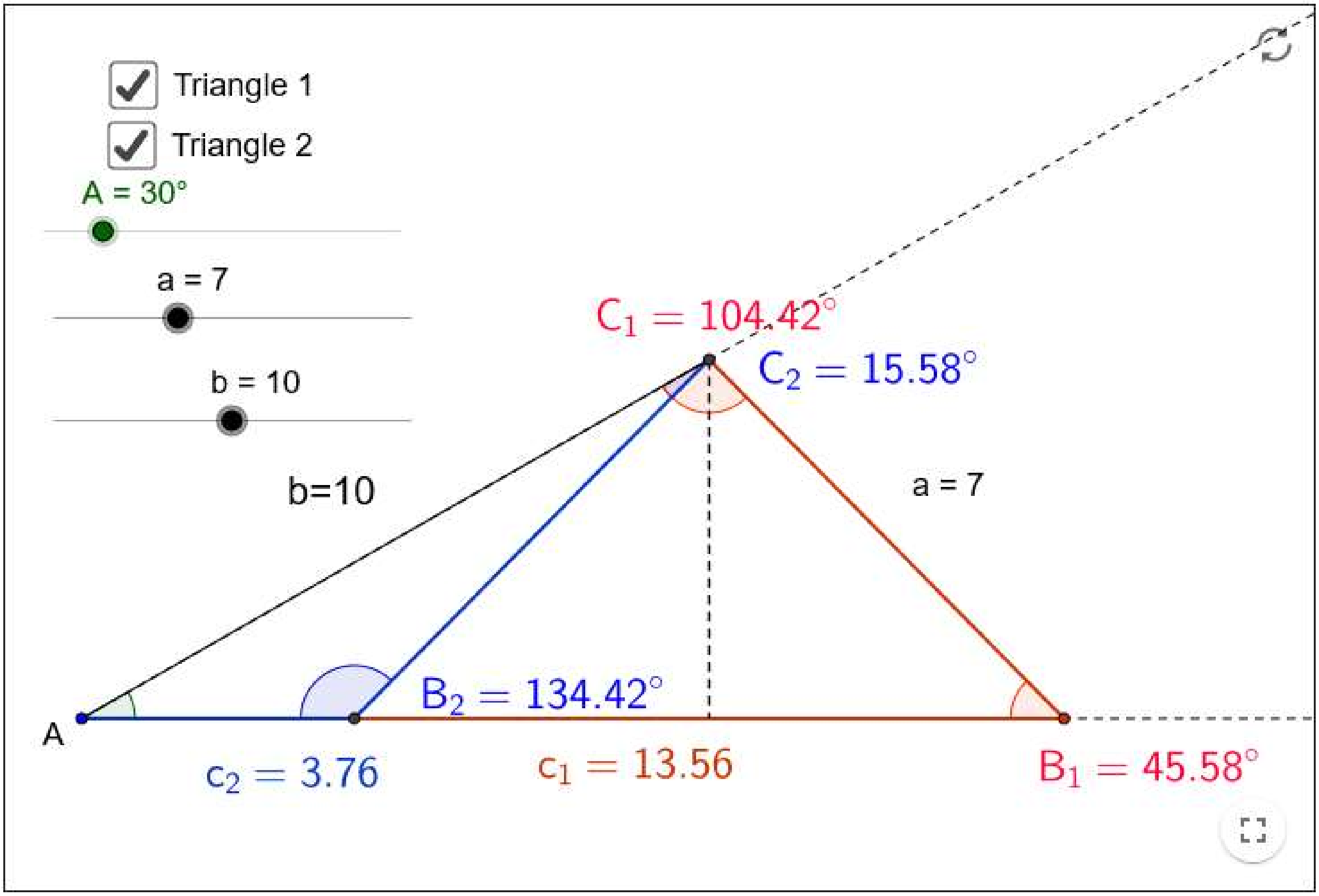
The calculator trick is to replace [B] with 180-[B] to get its new value. Then find angle [C] and store it. Finally find side c as before. Here's the calculator screen:



Summarizing the second solution gives

$$B' = 134.42°$$
$$C' = 15.585°$$
$$c' = 3.7613$$

Here's a picture of this problem from the SSA Illustrator [link above].



For these two triangles, the larger side c is 13.5592 and the smaller is 3.7613.