Trig Equations Part II: Approximate Solutions and Multiple Angles

In these notes we consider a hybrid trig equation involving the quadratic formula, and some example which have a multiple angle.

Example: The smallest positive number for which

$$4\cos^2 x - 9\cos x + 2 = 0$$

is $x = ?$

Solution: First we must recognize that this equation is a quadratic equation in $\cos x$ instead of x. To solve it for $\cos x$ we will use the quadratic formula with $a = 4, b = -9,$ and $c = 2.$

$$\begin{aligned}\cos x &= \frac{-(-9) \pm \sqrt{(-9)^2 - 4(4)(2)}}{2 \cdot 4} \\ &= \frac{9 \pm \sqrt{81 - 32}}{8} \\ &= \frac{9 \pm \sqrt{49}}{8} \\ &= \frac{9 \pm 7}{8} \\ &= 2 \text{ or } \frac{1}{4}\end{aligned}$$

[Note: The facts that $a, b,$ and c are integers and that the discriminant $b^2 - 4ac = 49$, a perfect square, tell us that the left side of the equation could be factored.]

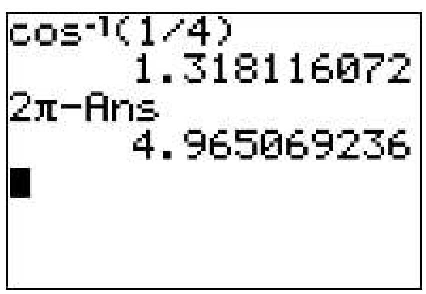
We now have two simple trig equations to solve

$$\left[\ \cos x = 2 \quad \text{or} \quad \cos x = \tfrac{1}{4}\ \right]$$

The first equation has **no solutions** since $\cos x \le 1$, always. Now consider

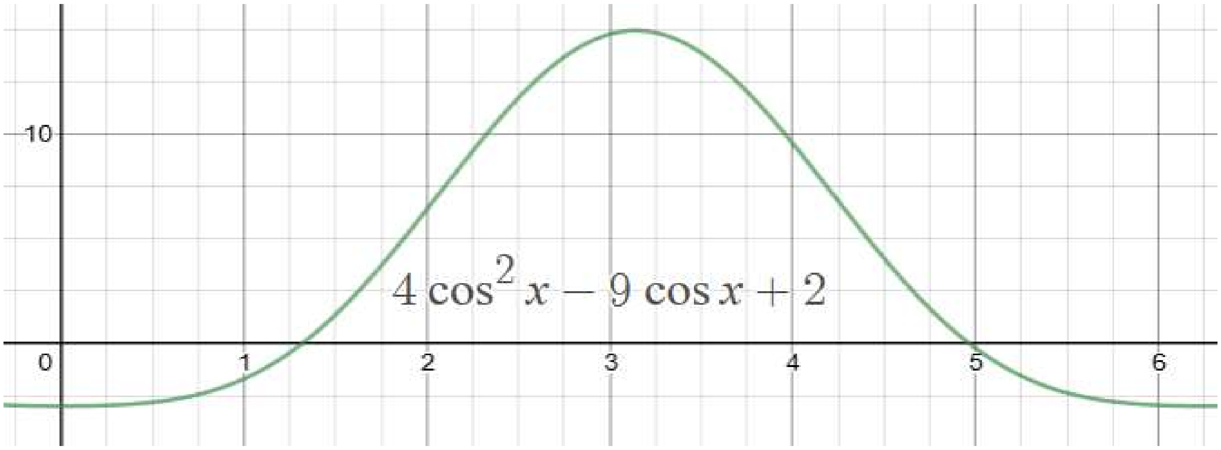
$$\cos x = \frac{1}{4}$$

Since the cosine function is positive in QI and QIV, there are two answers. Since this equation does not involve the special angles, we will use the calculator in **radian** mode to find the answer is QI. The answer in QI is also a reference angle for the answer in QIV, and so we will subtract is from 2π to get both approximate solutions in the interval $[0, 2\pi]$.



The smallest positive number is $x = 1.3181$.

Let's do a graph to see these two results. We will use Desmos since it recognizes the $\cos^2$ notation.



From the scale on the x-axis you can read the approximate solutions.

Example: The equation

$$cos(3x) = \frac{1}{2}$$

has six solutions between 0 and 360 degrees. Find the two solutions between 0 and 120 degrees.

Solution: We have degree mode and special angles in this problem. Since $\cos(3x)$ is positive, we must have the angle $3x$ in either QI or QIV. The reference angle in QI (and a solution) is $60°$. The angle ins QIV must be

$$360° - 60° = 300°$$

So far, we have

$$3x = 60°, 300°$$

We will divide by 3 to find x. Since we must have **all** answers between $0°$ and $360°$ we will add multiples of $360°$ to both $60°$ and $300°$.

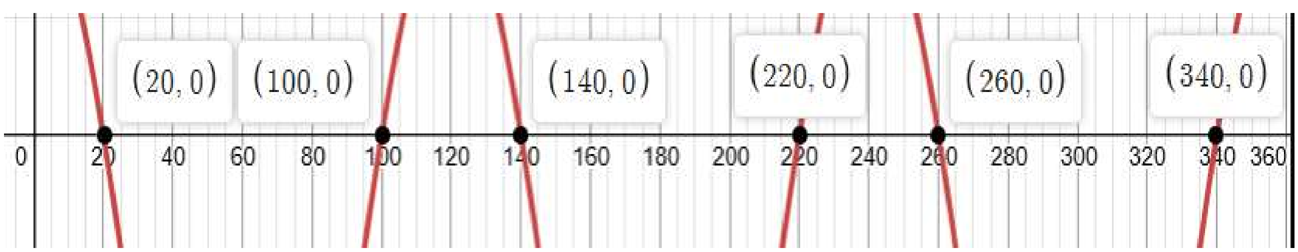
$$3x = 60°, 300°, 420°, 660°, 780°, 1020°$$

$$x = 20°, 100°, 140°, 220°, 260, °340°$$

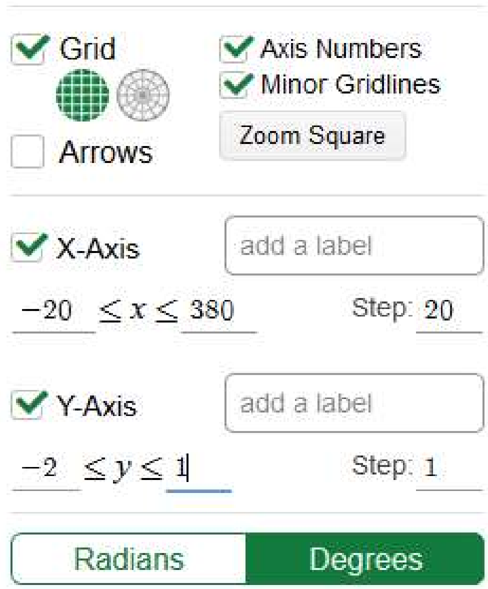
The angles between 0 and 120 degrees are $20°$ and $100°$. This time we'll graph the function

$$\cos(3x) - \frac{1}{2}$$

in **degree mode**! Here's the Desmos version.



Here's a snip showing the Desmos window settings:



You can also set this up with your calculator.

Example: The smallest positive number for which

$$3\sin(2x-1) = 1$$

is $x = ?$

Solution: We are in radian mode here.

$$\sin(2x-1) = \frac{1}{3}$$

Since 1/3 is positive, this angle must be in QI or QII. The inverse sine function will give us the solution (and reference angle) in QI, the smallest number possible.

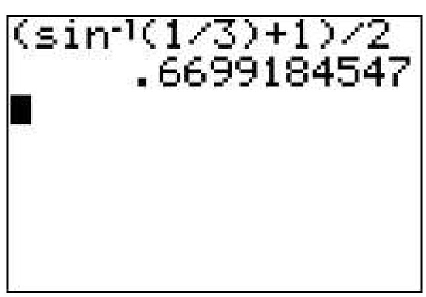
$$2x-1 = \sin^{-1}\left(\frac{1}{3}\right)$$

Now treat this as a linear equation in x. Add 1 and divide by 2.

$$x = \frac{\sin^{-1}\left(\frac{1}{3}\right)+1}{2}$$

$$= 0.66992$$

Here's the calculator view.



Extra innings: A graph of $3\sin(2x-1)-1$ shows that there are three more solutions between 0 and 2π. Can you find each rounded off to five decimal places?