Notes: Inverse Trig Functions and Examples

The sine, cosine, and tangent functions are periodic, and so we must restrict the domain in order to define an inverse that will be a function.

1. **Inverse Sine**, written $\arcsin(x)$ or $\sin^{-1}(x)$ [remember, the -1 is NOT an exponent].

$$y = \sin^{-1}(x) = \arcsin(x)$$

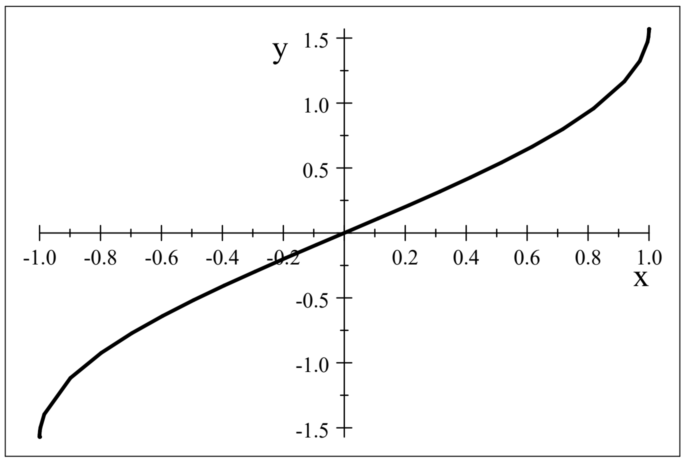
means

A. $\sin(y) = x$ and

B. $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ or in degree mode, $-90° \leq y \leq 90°$

"y is the number (angle) in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x."

Here's a graph of $\sin^{-1}(x)$. Note that $\frac{\pi}{2} = 1.57$, approximately



C. The domain of $\arcsin(x)$ is $[-1, 1]$ and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] = [-90°, 90°]$ (degree mode) .

D. $\sin^{-1}(x)$ is an odd function: $\sin^{-1}(-x) = -\sin^{-1}(x)$.

2. **Inverse Cosine**, written $\arccos(x)$ or $\cos^{-1}(x)$

$$y = \cos^{-1}(x) = \arcsin(x)$$

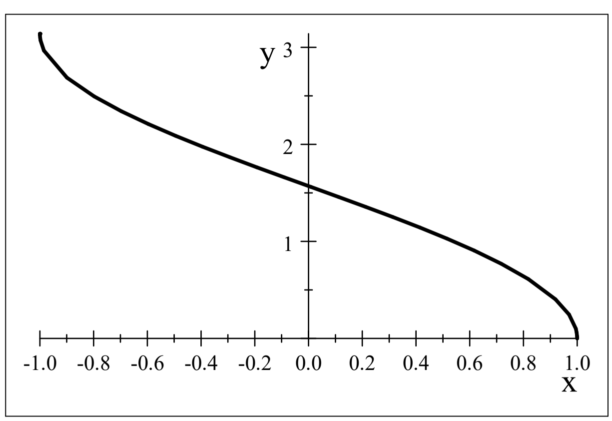
means

A. $\cos(y) = x$ and

B. $0 \leq y \leq \pi$ or in degree mode, $0° \leq y \leq 180°$

"y is the number (angle) in the interval $[0, \pi]$ whose cosine is x."

Here's a graph of $\arccos x$



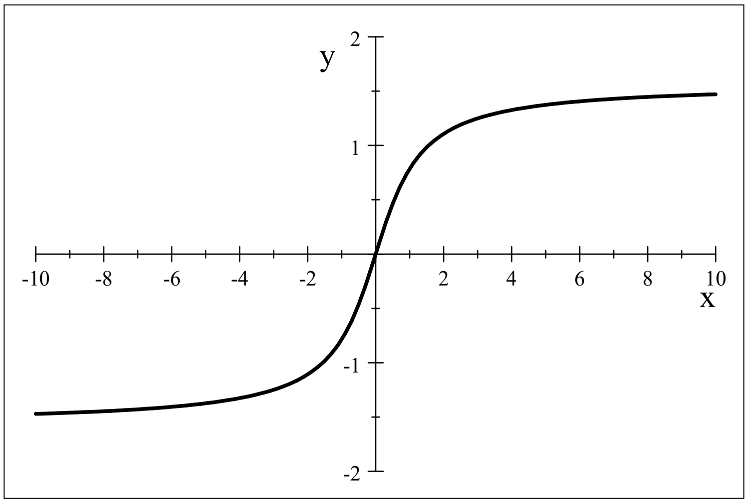
C. The domain of $\cos^{-1}(x)$ is $[-1, 1]$ and the range is $[0, \pi] = [0°, 180°]$.
D. $\cos^{-1}(x)$ is neither an odd function nor an even function.

3. **Inverse Tangent**, written $\arctan(x)$ or $\tan^{-1}(x)$

$$y = \tan^{-1}(x) = \arctan(x)$$

means

A. $\tan(y) = x$ and
B. $-\frac{\pi}{2} < y < \frac{\pi}{2}$ or $-90° < y < 90°$ (degree mode)
Here's a graph of $\arctan(x)$



C. $\tan^{-1}(x)$ is an odd function
D. The domain of $\arctan x$ is $(-\infty, \infty)$ and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The graph has horizontal asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$.

Examples

1. Inverse trig function values:

A. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4} = -45°$(degree mode)
B. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$ (not $-\frac{\pi}{4}$) $= 135°$

2. Complete the following:

A. $\arccos(\cos(37°)) = ?°$
The angle is in QI and so the answer is 37°
B. $\arccos(\cos(-25°)) = ?°$
The range of $\arccos(x)$ is $[0°, 180°]$ and so the answer can't be $-25°$. But the cosine function is **even**, so

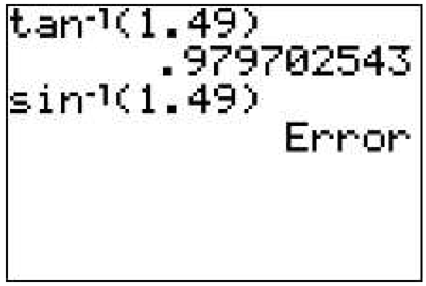
$$\arccos(\cos(-25°)) = \arccos(\cos(25°)) = 25°$$

C. $\arccos(\cos(100°)) = ?°$
Because the $100°$ is in the range of $\arccos(x)$, the answer is $100°$.
You can check these with your calculator in **Degree** mode.

3. Find an approximate value of each expression correct to at least five decimal places if it is defined, **otherwise**, **input undefined**.

A. $\tan^{1} 1.49 = ?$
B. $\sin^{-1} 1.49 = ?$

Be sure to use **Radian** mode on your calculator. Since 1.49 is not in the domain of $\sin^{-1}(x)$ it will be **undefined**.



4. Find the exact value of each expression:

A. $\sec(\arctan(2))$

B. $\cos\left(2\sin^{-1}\left(\frac{5}{13}\right)\right)$

For part A, let angle $A = \arctan 2$ so that $\tan(A) = 2 = \frac{2}{1}$. Since tangent is $\frac{y}{x}$ we have

$$x = 1$$
$$y = 2$$
$$r = \sqrt{1^2 + 2^2} = \sqrt{5}$$

Then

$$\begin{aligned}\sec(\arctan(2)) &= \sec(A)\\ &= \frac{r}{x}\\ &= \frac{\sqrt{5}}{1}\\ &= \sqrt{5}\end{aligned}$$

For part B, let $A = \sin^{-1}\frac{5}{13}$ and so the form of the problem is $\cos(2A)$. We have a choice of two formulas. Use the one with sine since $\sin(A) = \frac{5}{13}$ is given.

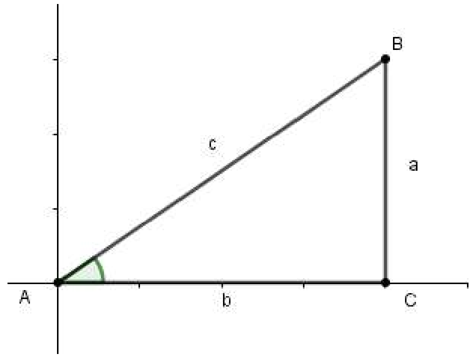
$$\begin{aligned}\cos(2A) &= 1 - 2\sin^2 A\\ &= 1 - 2\left(\frac{5}{13}\right)^2\\ &= \frac{119}{169}\end{aligned}$$

5. Complete the identity using the triangle method

A. $\cos(\tan^{-1}(x)) = ?$

B. $\sin(\sec^{-1}(x)) = ?$

For **parts A** and **B** use this standard right triangle



Let $A = \tan^{-1}(x)$ so that $\tan(A) = x = \frac{x}{1} = \frac{opp}{adj} = \frac{b}{a}$. In the picture $a = x,\ b = 1,$ and $c = \sqrt{x^2 + 1^2}$. The

$$\begin{aligned}\cos(\tan^{-1}(x)) &= \cos(A) \\ &= \frac{adj}{hyp} \\ &= \frac{b}{c} \\ &= \frac{1}{\sqrt{x^2+1}}\end{aligned}$$

For an online answer, type

1/sqrt(x^2+1)

For **part B** consider the same triangle with $A = \sec^{-1}(x)$ and

$$\sec(A) = x = \frac{x}{1} = \frac{hyp}{adj} = \frac{c}{b}$$

So $a = ?,\ b = 1$ and $c = x$. Using the Pythagorean Theorem gives

$$\begin{aligned}a &= \sqrt{c^2 - b^2} \\ &= \sqrt{x^2 - 1^2}\end{aligned}$$

Then

$$\begin{aligned}\sin(\sec^{-1}(x)) &= \sin(A) \\ &= \frac{opp}{hyp} \\ &= \frac{a}{c} \\ &= \frac{\sqrt{x^2-1}}{x}\end{aligned}$$

The online answer form is

sqrt(x^2-1)/x