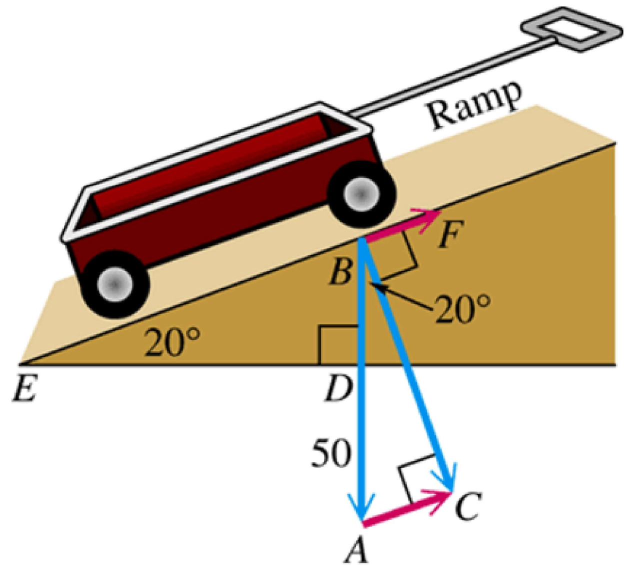
Notes on Vectors: Applications

In the following examples the vectors will be used to represent forces or displacements. Our first example is a typical inclined plane situation, pictured here:



[This illustration from the Math 1316 textbook Trigonometry, 11th Edition, authors Margaret Lial, John Hornsby, David I. Schneider, Callie Daniels. Published by Pearson]

The weight, 50 pounds, is represented by vector $\overrightarrow{BA}$ and points straight down. From the geometry here we see that the angle of inclination of the ramp is equal to angle B. Since angle C is a right angle, the magnitude of the vector $\overrightarrow{AC}$ is given by

$$\left|\overrightarrow{AC}\right| = \left|\overrightarrow{BA}\right| \sin 20^\circ$$

Another notation for this formula is

$$F_{\parallel} = w \sin\theta$$

where $F_{\parallel}$ is the magnitude of the force parallel to the plane, w is the magnitude of the weight vector, and θ is the angle of inclination of the ramp.

When doing friction calculations with a block on an inclined plane, we need to find the magnitude of vector $\overrightarrow{BC}$, a vector which is perpendicular to the plane. Then we have

$$F_{\perp} = w \cos\theta$$

Example 1: Calculate the force (in Newtons) required to push a 40kg wagon up a 0.3 radian inclined plane. One Newton (N) is equal to 1kg•m/s^2, and the force due to gravity on the wagon is F=m*g, where m is the mass of the wagon, and g is the acceleration due to gravity (9.8m/s^2). Ignore friction in this problem.

Solution: All we need to find here is

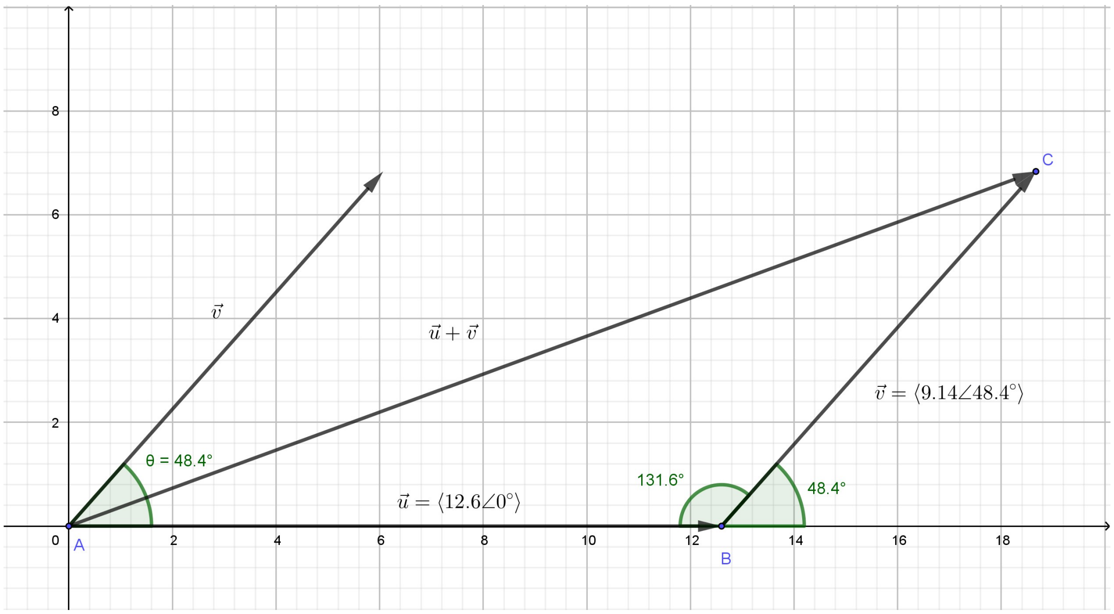
$$\begin{aligned} F_{\parallel} &= w \sin\theta \\ &= 40 * 9.8 \sin(0.3) \quad \text{(radian mode)} \\ &= 115.8439 \end{aligned}$$

The exercise has

Force = $\boxed{115.8439}$ N

Example 2 Two forces act on an object. The first force has a magnitude of 12.6 pounds. The second force has a magnitude of 9.14 pounds. The angle between the two forces is 48.4 degrees. The magnitude of the resultant is ? Round off your answer to at least three decimal places.

Solution: Since there are two vectors, we can draw a tip-to-tail triangle picture. We will assign the first vector a direction angle of $0°$ with a magnitude of 12.6 pounds (U.S. common system). The second vector will have magnitude 9.14 pounds and direction angle $48.4°$. Here's a picture.



In the picture the first force $\vec{u}$ is

$$\vec{u} = \langle 12.9\angle 0°\rangle$$

and the second $\vec{v}$ is

$$\vec{v} = \langle 9.14\angle 48.4°\rangle$$

To add two vectors in polar form we can use one of two methods:

(1) a geometric method in which we solve the triangle for the side which represents $\vec{u}+\vec{v}$ (side b in the picture). We have a SAS situation and can use the Law of Cosines.

(2) an analytic method in which the vectors are converted to rectangular form, added, and then converted back to polar form.

Since we have to find the length of only one side, we will use the geometric method and the Law of Cosines (noting that the known angle is $180° - 48.4°$)

$$\begin{aligned} \left|\vec{u}+\vec{v}\right|^2 &= 12.6^2 + 9.14^2 - 2(12.6)(9.14)\cos 131.6° \\ &= 395.22 \quad \text{(rounded off)} \end{aligned}$$

and

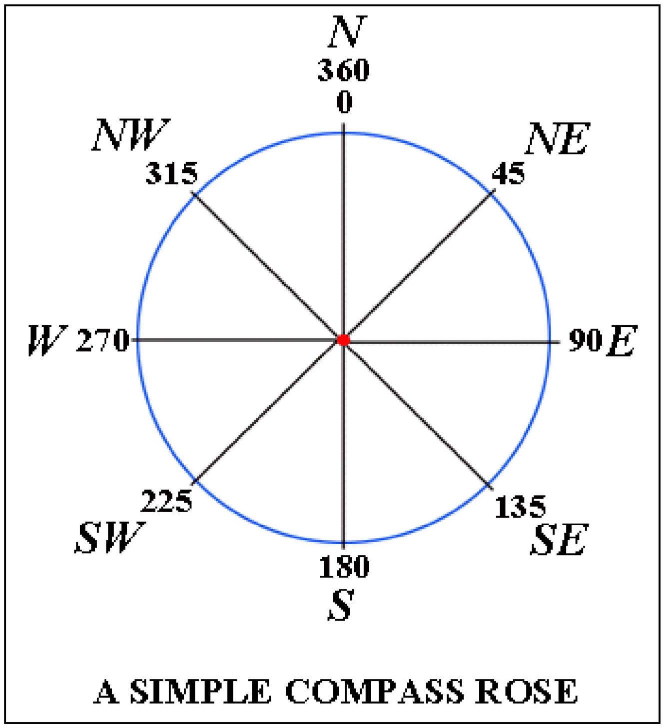
$$\left|\vec{u}+\vec{v}\right| = 19.8802$$

Later we will see that in a situation with three or more vectors, the analytic approach will be simpler.

Example 3: A boat is heading due east at 20 km/hr (relative to the water). The current is moving toward the southwest at 6 km/hr.

1. **A**. Give the vector representing the actual movement of the boat.

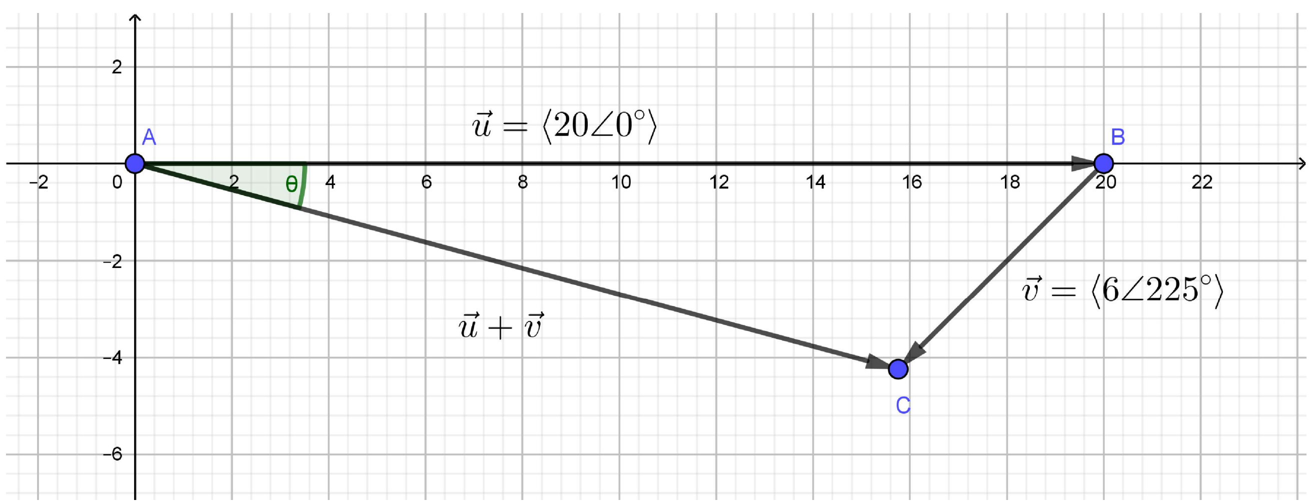
Solution: Here's a picture of compass directions. North is always towards the top of the map.



Compass angles are measured clockwise from North. To represent the vectors involved, we must covert compass angles to x-y coordinate system angles (measured counter clockwise from the positive x-axis). The direction angle for the boat is $0°$, and the direction angle for the current is $225°$ (this is the only compass angles which have the same direction angle are $45°$ and $225°$). The vector we wish to find is

$$\langle 20\angle 0°\rangle + \langle 6\angle 225°\rangle$$

Here's a picture of the displacement vector situation in a tip-to-tail diagram.



We now convert $\vec{u}$ and $\vec{v}$ to rectangular form and add the x-components and the y-components. Here's a table to help organize this work.

Vector	$r\cos\theta$	$r\sin\theta$
$\langle 20\angle 0°\rangle$	20.0000	0.0000
$\langle 6\angle 225°\rangle$	-4.2426	-4.2426
Sum	15.7574	-4.2426

The vector that answers question A is

$$\langle 15.7574, -4.2426\rangle$$

B. How fast is the boat moving, relative to the ground?

Solution: To answer this we find the magnitude (length) of $\vec{u} + \vec{v}$

$$\begin{aligned}|\vec{u} + \vec{v}| &= \sqrt{(15.7574)^2 + (-4.2426)^2} \\ &= 16.3185\end{aligned}$$

C. By what angle does the current push the boat off its due east course? Your answer should be a positive angle less than 180 **degrees** (emphasis mine to show degree mode).

Solution: We will find the direction angle for $\vec{u} + \vec{v} = \langle 15.7574, -4.2426\rangle$. Since this angle is in QIV, the inverse tangent is negative (ASTC). Because the problem doesn't ask for a compass bearing, the absolute value of the angle will be the solution.

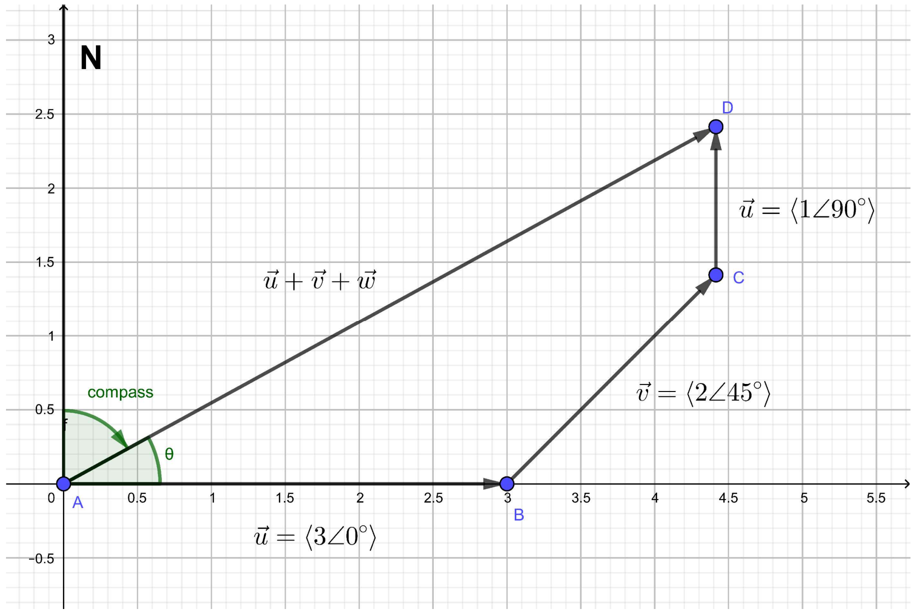
$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{-4.2426}{15.7574}\right) \\ &= -15.0694°\end{aligned}$$

The answer for part C is

$$15.0694°$$

Example 4: You leave your friend behind on the shore and you travel 3 miles due east in your boat. Then you travel 2 miles northeast. Then you travel 1 mile due north. Your friend can see you at a distance of [ ] miles and at a bearing of [ ] degrees. Enter your answers with at least 3 digits beyond the decimal point.

Solution: Here's a diagram of the three displacement vectors $\langle 3\angle 0°\rangle$, $\langle 2\angle 45°\rangle$, and $\langle 1\angle 90°\rangle$ in tip-to-tail form. Note that the direction angle $45°$ is the same as the compass angle.



We will convert the vectors to rectangular form and add the components. This will give us the sum vector in rectangular form. We can then find the magnitude and direction angle θ. Finally, we convert the direction angle to a compass bearing. Heres a table

Vector	$r\cos\theta$	$r\sin\theta$
$\langle 3\angle 0^\circ\rangle$	3.0000	0.0000
$\langle 2\angle 45^\circ\rangle$	1.4142	1.4142
$\langle 1\angle 90^\circ\rangle$	0.0000	1.0000
Sum	4.4142	2.4142

We now convert the vector

$$\vec{u}+\vec{v}+\vec{w} = \langle 4.4142, 2.4142\rangle$$

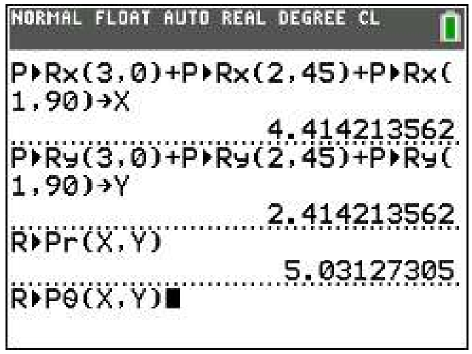
to polar form. The angle is in QI, and so the inverse tangent will be correct

$$\begin{aligned}\vec{u}+\vec{v}+\vec{w} &= \sqrt{4.4142^2+2.4142^2}\\ &= 5.0313\\ \theta &= \tan^{-1}\left(\frac{2.4142}{4.4142}\right)\\ &= 28.6751^\circ\end{aligned}$$

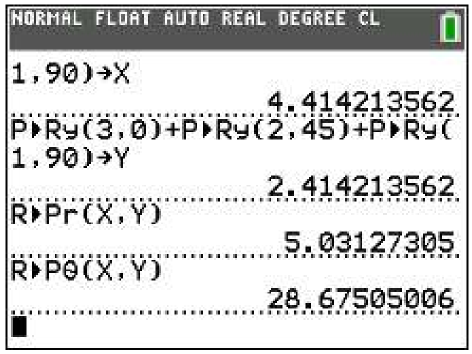
Finally, convert the direction angle to a compass angle by subtracting θ from 90°.

$$\text{bearing} = 61.3249^\circ$$

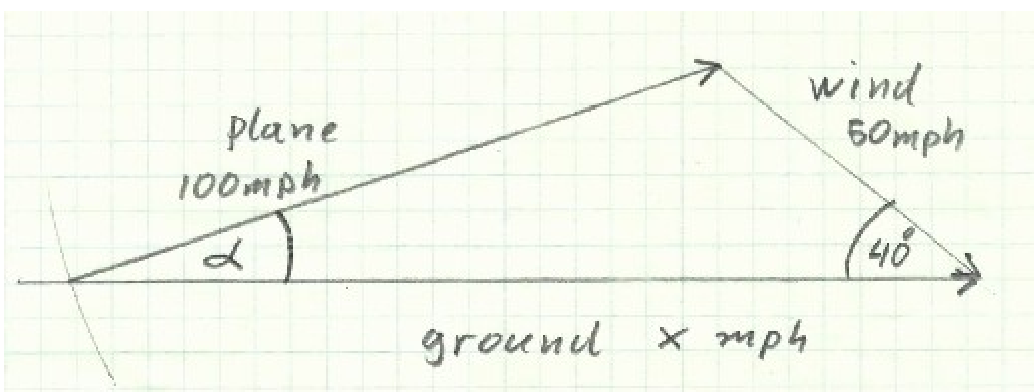
Major calculator trick: You can use the ANGLE menu on the calculator to get these results. We will store the sum components in X and Y (<ALPHA><1>).



and then finish with <ENTER>



Example 5: Calculating the necessary aircraft heading to counter a wind velocity and proceed along a desired bearing to a destination is a classic problem in aircraft navigation. It makes good use of the law of sines and the law of cosines. Suppose you wish to fly in a certain direction relative to the ground. The wind is blowing at 50mph at an angle of 40 degrees to that direction. Your plane is flying at 100mph with respect to the surrounding air. The situation is illustrated in this Figure (where your desired direction of travel is due East):



Then you head into the wind at an angle of degrees (enter your value of α), and your ground speed is miles per hour (enter your value of x).

Solution: In this problem we have the SSA situation we saw in the Law of Sines examples. In particular, the opposite side of the angle is greater than the adjacent sides, and so there is only one triangle possible (no ambiguous case here!) First we find angle α. We will not need to use the Law of Cosines.

$$\frac{50}{\sin\alpha} = \frac{100}{\sin 40°}$$
$$\sin\alpha = \frac{50\sin 40°}{10}$$
$$\alpha = \sin^{-1}\left(\frac{50\sin 40°}{100}\right)$$
$$= 18.7472°$$

The angle opposite side x is

$$180° - 40° - 18.7472° = 121.2528°$$

Now we use the Law of Sines again to find x.

$$\frac{x}{\sin 121.2528°} = \frac{100}{\sin 40°}$$
$$x = \frac{100\sin 121.2528°}{\sin 40°}$$
$$= 132.9967$$

Summarizing,

angle $\alpha = 18.7472°$ and the ground speed $x = 132.9967$ miles/hour

Comment

Many problems in physics and engineering can be written with vector equations. There is one more vector operation not covered in these notes: the **cross product**, an operation between two three-dimensional vectors. Cross product is used in applications involving **torque**, particularly in statics problems.