Notes: Examples of Vector Operations 2 - The Dot Product

In the previous notes we covered examples of vector conversions, addition/subtraction, and scalar multiplication. Here we add one more operation to our vector toolkit: the dot product of two vectors. If $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$ then

$$\mathbf{u} \cdot \mathbf{v} = \langle a, b \rangle \cdot \langle c, d \rangle = ac + bd$$

The dot product of two vectors is a real number which may be positive, zero, or negative.

In **physics**, if a force vector $\vec{F}$ does work on an object and moves it through a displacement vector $\vec{d}$ then the work done is the dot product

$$W = \vec{F} \cdot \vec{d}$$

In many problems the force moves the object in the same direction, and then the work done is simply the magnitude of the force times the magnitude of the displacement.

Geometrically, the dot product measures the tendency of two vectors to point in the same direction. Here's the formula:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$$

where θ is the angle between the vectors **u** and **v** with $0° \leq \theta \leq 180°$. Then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

The denominator $|\mathbf{u}||\mathbf{v}|$ is always positive. If

1. $\mathbf{u} \cdot \mathbf{v} > 0$, then $\cos\theta > 0$, and θ is in QI.
2. $\mathbf{u} \cdot \mathbf{v} = 0$, then $\cos\theta = 0$, and $\theta = 90°$ (or $\frac{\pi}{2}$). The vectors are said to be **orthogonal**.
3. $\mathbf{u} \cdot \mathbf{v} > 0$, then $\cos\theta < 0$, and θ is in QII.

If the vectors point in the **same direction**, then the angle between them is $0°$. Then $\cos 0° = 1$ and

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|$$

In particular, the dot product of a vector with itself is the square of the magnitude:

$$\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}||\mathbf{u}| = |\mathbf{u}|^2$$

If the vectors point in the **opposite direction**, then the angle between them is $180°$. Then $\cos 180° = -1$ and

$$\mathbf{u} \cdot \mathbf{v} = -|\mathbf{u}||\mathbf{v}|$$

For an interactive illustrator of the dot product go to

https://www.geogebra.org/m/KMifob4i

Examples

Example 1: Let $\mathbf{u} = \langle -2, -1 \rangle$ and $\mathbf{v} = \langle 1, 5 \rangle$ Find the following:

1. **A**. $\mathbf{u} \cdot \mathbf{v} = ?$

Solution:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \langle -2, -1 \rangle \cdot \langle 1, 5 \rangle \\ &= (-2)(1) + (-1)(5) \\ &= -7 \end{aligned}$$

B. $\|\mathbf{v}\| = ?$ (in some textbooks double vertical lines are used for vectors instead of single vertical lines)

Solution:

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{(1)^2 + (5)^2} \\ &= \sqrt{26} \end{aligned}$$

Example 2: Find $\mathbf{a} \cdot \mathbf{b}$ if $|\mathbf{a}| = 3$, $|\mathbf{b}| = 9$, and the angle between **a** and **b** is $-\pi/7$ radians.

Solution:

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= 3 \cdot 7 \cos\left(-\frac{\pi}{7}\right) \quad \text{(radian mode)} \\ &= 18.9203 \end{aligned}$$

Example 3:

A. Find a unit vector $\vec{u}$ from the point $P = (2, 3)$ and toward the point $Q = (5, 7)$.

Solution: A **unit vector** is a vector whose length is 1. To construct this vector we first build vector $\overrightarrow{PQ}$

$$\overrightarrow{PQ} = \langle 5 - 2, 7 - 3 \rangle = \langle 3, 4 \rangle$$

Then find the length of $\overrightarrow{PQ}$

$$\begin{aligned} \left|\overrightarrow{PQ}\right| &= |\langle 3, 4 \rangle| \\ &= \sqrt{3^2 + 4^2} \\ &= 5 \end{aligned}$$

Then multiply $\overrightarrow{PQ}$ by the reciprocal of the length: $\frac{1}{5}$

$$\frac{1}{5}\langle 3, 4 \rangle = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

Finally check the length of $\langle \frac{3}{5}, \frac{4}{5} \rangle$:

$$\left|\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle\right| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$$
$$= \sqrt{\frac{9}{25} + \frac{16}{25}}$$
$$= \sqrt{\frac{25}{25}}$$
$$= 1$$

The vector $\langle \frac{3}{5}, \frac{4}{5} \rangle$ has the correct length and is what we're looking for

$$\vec{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

B. Find a vector $\vec{v}$ of length 10 pointing in the same direction.

Solution: To construct $\vec{v}$ we need only to multiply the unit vector $\vec{u}$ by 10.

$$10\vec{u} = 10\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = \langle 6, 8 \rangle$$

It is easy to check that $\vec{v} = \langle 6, 8 \rangle$ has length 10.

Online homework note: this answer can be written with $\vec{i}$, $\vec{j}$ notation. With $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$

$$\vec{v} = 6\vec{i} + 8\vec{j}$$

Many textbooks use a "hat" to indicate that we have unit vectors.

$$\vec{v} = 6\hat{i} + 8\hat{j}$$

In our online homework you can type

3/5i+4/5j

without worry about arrows or hats and the system will preview it as:

$$\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$$

and it will check correctly.

Example 4: Consider the three points:

$A = (1, 4)$
$B = (5, 5)$
$C = (6, 7)$

Determine the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Solution: First calculate the two vectors:

$$\overrightarrow{AB} = \langle 5 - 1, 5 - 4 \rangle = \langle 4, 1 \rangle$$
$$\overrightarrow{AC} = \langle 6 - 1, 7 - 4 \rangle = \langle 5, 3 \rangle$$

Next calculate the dot product

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = (4)(5) + (1)(3) = 23$$

Now find the magnitudes (lengths)

$$\overrightarrow{AB} = |\langle 4, 1\rangle| = \sqrt{17}$$

$$\overrightarrow{AC} = |\langle 5, 3\rangle| = \sqrt{34}$$

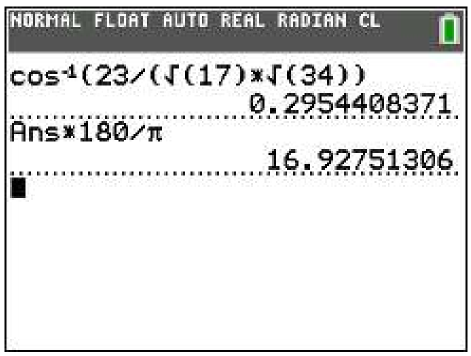
To finish, find $\cos\theta$

$$\cos\theta = \frac{23}{\sqrt{17}\sqrt{34}}$$

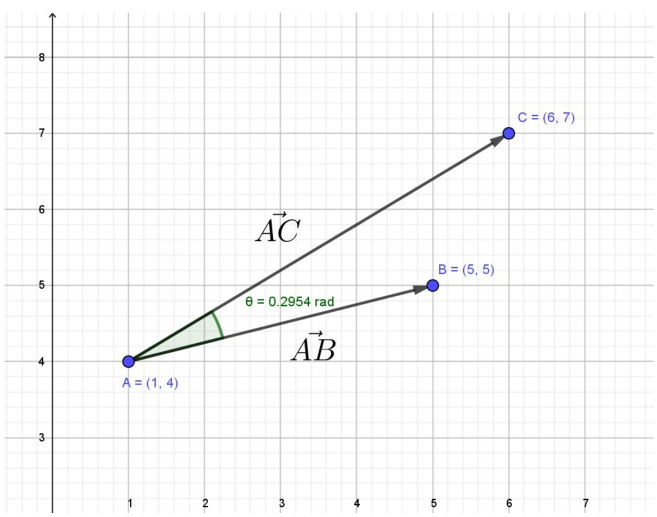
$$\theta = \cos^{-1}\left(\frac{23}{\sqrt{17}\sqrt{34}}\right)$$

$$= 0.2954 \text{ radians}$$

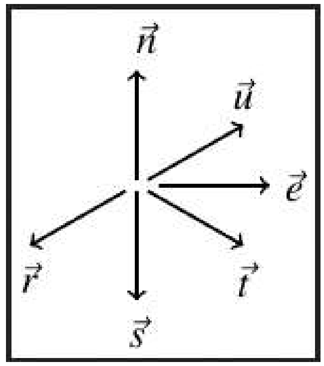
Since the problem doesn't mention degrees anywhere, the answer is in radians. Remember, without the word "degree" or a ° symbol, it's radian mode.



The angle is about 17°. Here's a picture from GeoGebra.



Example 5: Using the geometric definition of the dot product, are the following dot products **positive**, **negative**, **or zero**? You may assume that angles that look the same are the same.



1. **A**. $\vec{s} \cdot \vec{t}$? **Answer**: Positive–the angle between them is less than $90°$.

B. $\vec{n} \cdot \vec{e}$? **Answer**: Zero–the angle between them is $90°$.

C. $\vec{e} \cdot \vec{r}$? **Answer**: Negative–the angle between them is greater than $90°$.