# Combining Functions

New functions can be formed from old ones in two basic ways:

**1. By arithmetic operations**

**A.** Addition

$$(f+g)(x) = f(x) + g(x)$$

**B.** Subtraction

$$(f-g)(x) = f(x) - g(x)$$

**C.** Multiplication

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

**D.** Division

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \text{ provided } g(x) \neq 0$$

Here are some examples

1. Let $f(x) = x^2 + 3$ and $g(x) = 2x - 4.$ Then

$$\begin{aligned}(f-g)(x) &= f(x) - g(x) \\ &= (x^2+3) - (2x-4) \\ &= x^2 - 2x + 7\end{aligned}$$

2. With the same definitions of $f, g$ we have

$$\frac{f}{g}(4) = \frac{f(4)}{g(4)} = \frac{4^2+3}{2 \cdot 4 - 4} = \frac{19}{4}$$

**2. By composition**.

If $f, g$ are functions with *ran f* a subset of *dom g* then the *composition of f with g* is defined by

$$(f \circ g)(x) = f(g(x))$$

Examples

1. Let $f(x) = x^2 + 3$ and $g(x) = 2x - 4.$ Then

$$\begin{aligned}f \circ g(5) &= f(g(5)) \\ &= f(2 \cdot 5 - 4) \\ &= f(6) \\ &= 6^2 + 3 \\ &= 39\end{aligned}$$

2. Using these same functions again we have

$$\begin{aligned}g \circ f(5) &= g(f(5)) \\ &= g(5^2 + 3) \\ &= g(28) \\ &= 2 \cdot 28 - 4 \\ &= 52\end{aligned}$$

Note that in these two examples $f \circ g(5) \neq g \circ f(5)$ and therefore composition of functions is **not commutative**:

$$f \circ g \neq g \circ f$$

In general

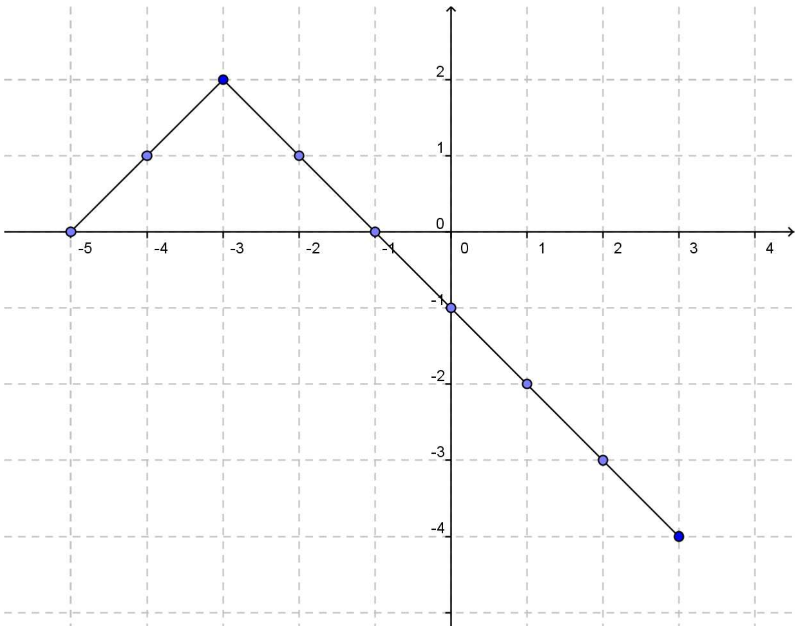
$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f(2x - 4) \\ &= (2x - 4)^2 + 3 \\ &= 4x^2 - 16x + 19 \end{aligned}$$

but

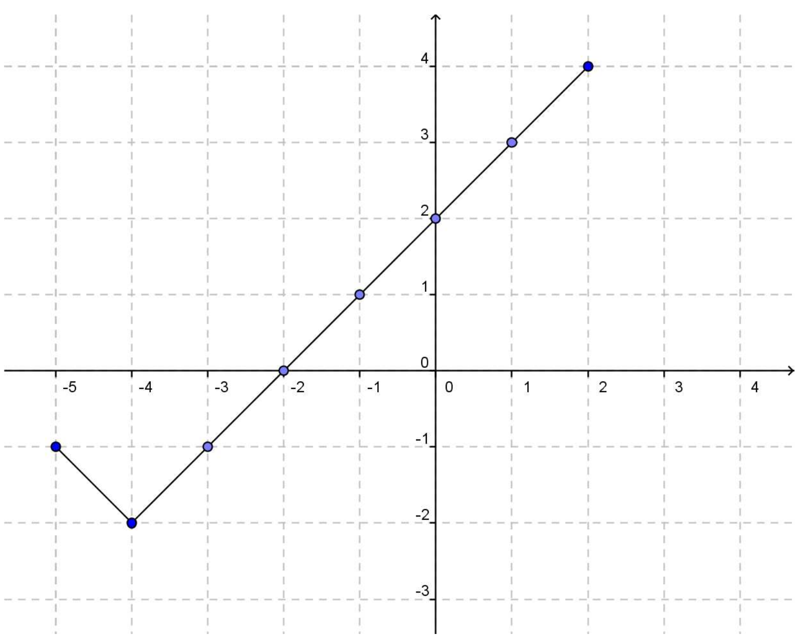
$$\begin{aligned} g \circ f(x) &= 2(x^2 + 3) - 4 \\ &= 2x^2 + 2 \end{aligned}$$

**Composition of Functions Defined Graphically**

If functions *f* and *g* are described graphically, function composition can be done if the graphs are clearly scaled.




function f




function g

**Example**: Using the graphs above, find $f \circ g(1)$.

Solution

$$\begin{aligned} f \circ g(1) &= f(g(1)) \\ &= f(3) \\ &= -4 \end{aligned}$$

**Composition of Functions defined by a Table**

In some cases, functions $f$ and $g$ may be given by a table. The composition of $f$ with $g$, $f \circ g(x)$ will be defined provided that the range of $g$ (the **inside** function) is a subset of the domain of $f$ (the **outside** function).

**Example**: Here's a table which defines two functions, $f(x)$ and $g(x)$.

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $f(x)$ | 3 | 1 | 5 | 2 | 4 |
| $g(x)$ | 5 | 3 | 4 | 2 | 1 |

a. $f \circ g(3) = f(g(3)) = f(4) = 2$

b. $g \circ f(3) = g(f(3)) = g(5) = 1$

**Decomposition of Functions**

Many of the functions we have seen can be decomposed into a composition of simpler functions. For example, if

$$h(x) = \sqrt{2x+1}$$

then $h = f \circ g$ where

$$g(x) = 2x + 1 \text{ (the "\textbf{inside}" function) and}$$

$$f(x) = \sqrt{x} \text{ (the "\textbf{outside}" function)}$$

Note that the decomposition of functions may have more than one result. In this example, we could also have

$$g(x) = 2x \text{ and}$$

$$f(x) = \sqrt{x+1}$$