Examples of Using and Verifying Identities Chapter 5 Sections 1 and 2

In the following examples, we will be using the fundamental identities on page 191.

Example: Find $\sin s$ if $\cos s = \frac{8}{17}$. and s is in quadrant IV.

Solution: Use the identity

$$\cos^2 s + \sin^2 s = 1$$
$$\left(\frac{8}{17}\right)^2 + \sin^2 s = 1$$
$$\sin^2 s = 1 - \left(\frac{8}{17}\right)^2$$
$$\sin^2 s = \frac{225}{289}$$

Since s is in QIV, $\sin s < 0$, and so

$$\sin s = -\sqrt{\frac{225}{289}}$$
$$\sin s = -\frac{15}{17}$$

Example: Find $\sin\theta$, given that $\tan\theta = -\frac{\sqrt{3}}{5}$, $\cos\theta < 0$

Solution: Start with the identity

$$\tan^2\theta + 1 = \sec^2\theta$$
$$\left(-\frac{\sqrt{3}}{5}\right)^2 + 1 = \sec^2\theta$$
$$\frac{3}{25} + 1 = \sec^2\theta$$
$$\frac{28}{25} = \sec^2\theta$$

Since $\sec\theta = \frac{1}{\cos\theta}$, we have

$$\frac{1}{\cos^2\theta} = \frac{28}{25}$$
$$\cos^2\theta = \frac{25}{28}$$

Finally using the identity $\cos^2 s + \sin^2 s = 1$

$$\frac{25}{28} + \sin^2\theta = 1$$
$$\sin^2\theta = 1 - \frac{25}{28}$$
$$\sin^2\theta = \frac{3}{28}$$

Because tangent and cosine are negative, θ is in QII and $\sin\theta > 0$:

$$\sin\theta = \sqrt{\frac{3}{28}}$$

Simplifying the radicals gives

$$\sin\theta = \frac{3\sqrt{7}}{14}$$

Example: Write the expression in terms of sine and cosine and then simplify

$$-\sin^2\theta(\csc^2\theta - 1)$$

Solution: $\csc\theta = \dfrac{1}{\sin\theta}$ and so we have

$$\begin{aligned}
-\sin^2\theta(\csc^2\theta - 1) &= -\sin^2\theta\left(\frac{1}{\sin^2\theta} - 1\right) \\
&= -1 + \sin^2\theta \\
&= -(\sin^2\theta + \cos^2\theta) + \sin^2\theta \\
&= -\sin^2\theta - \cos^2\theta + \sin^2\theta \\
&= -\cos^2\theta
\end{aligned}$$

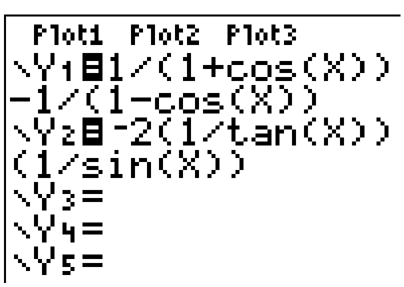
Example: Simplify the expression

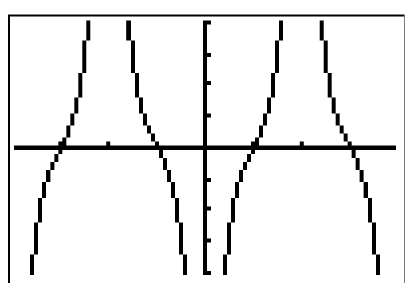
$$\frac{1}{1+\cos x} - \frac{1}{1-\cos x}$$

Solution: The common denominator is $(1+\cos x)(1-\cos x)$ so that we can write the expression as

$$\begin{aligned}
\frac{1}{1+\cos x} \cdot \frac{1-\cos x}{1-\cos x} - \frac{1}{1-\cos x} \cdot \frac{1+\cos x}{1+\cos x} &= \frac{1-\cos x}{1-\cos^2 x} - \frac{1+\cos x}{1-\cos^2 x} \\
&= \frac{1-\cos x - 1 - \cos x}{1-\cos^2 x} \\
&= \frac{-2\cos x}{1-\cos^2 x} \\
&= \frac{-2\cos x}{\sin^2 x} \\
&= -2\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} \\
&= -2\cot x \csc x
\end{aligned}$$

Remark: We can visualize the identity by graphing $\mathbf{Y}_1 = \dfrac{1}{1+\cos x} - \dfrac{1}{1-\cos x}$ and $\mathbf{Y}_2 = -2\cot x \csc x$ and seeing that they produce identical graphs





Example: Verify the identity, showing all steps:

$$\cos x - \sec x = -\sin x \tan x$$

Solution: In general, proving an identity involves transforming the expression on one side of the equation to the other side. In some cases, we may work on both sides of the equation until we come to a common expression. In this example we start with the left side

$$\begin{aligned}
\cos x - \sec x &= \cos x - \frac{1}{\cos x} && \text{reciprocal identity}\\
&= \frac{\cos^2 x}{\cos x} - \frac{1}{\cos x} && \text{common denominator}\\
&= \frac{\cos^2 x - 1}{\cos x} && \text{subtract fractions}\\
&= \frac{\cos^2 x - (\cos^2 x + \sin^2 x)}{\cos x} && \text{Pythagorean identity}\\
&= \frac{-\sin^2 x}{\cos x} && \text{simplify numerator}\\
&= -\frac{\sin x}{1} \cdot \frac{\sin x}{\cos x} && \text{expand fraction product}\\
&= -\sin x \tan x && \text{quotient identity.}
\end{aligned}$$

Since the left side is equal to the right, the equation is an identity.

Example Verify the identity, showing all steps:

$$\frac{\sin x - 1}{\sin x + 1} - \frac{\sin x + 1}{\sin x - 1} = 4\tan x \sec x$$

Solution: Again, we begin with the left side, getting a common denominator of $(\sin x + 1)(\sin x - 1)$

$$\begin{aligned}
\frac{\sin x - 1}{\sin x + 1} \cdot \frac{\sin x - 1}{\sin x - 1} - \frac{\sin x + 1}{\sin x - 1} \cdot \frac{\sin x + 1}{\sin x + 1} &= \frac{\sin^2 x - 2\sin x + 1}{1 - \sin^2 x} - \frac{\sin^2 x + 2\sin x + 1}{1 - \sin^2 x} && \text{expand fractions to co}\\
&= \frac{-4\sin x}{1 - \sin^2 x} && \text{combine numerators}\\
&= \frac{-4\sin x}{\cos^2 x} && \text{Pythagorean identity}\\
&= -4\frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} && \text{expand fraction product}\\
&= -4\tan x \sec x && \text{fundamental identities}
\end{aligned}$$

Since the left side is equal to the right, the equation is an identity.

Look at Example 5 in the text/etext on page 200. In this problem, each side of the equation is reduced to a simpler expression involving only sines and cosines.