Inverses of Relations and Functions

Definition: If f is a function then the *inverse of f*, written f^{-1}, is the function obtained by "reversing" the rule of function f.

For instance, if $f(x) = x + 5$ then f takes an input x and adds 5 to it to produce an ouput. To "undo" this, we must subtract 5: $f^{-1}(x) = x - 5$.

Similarly, the doubling function

$$f(x) = 2x$$

is reversed by the "halving" function

$$f^{-1}(x) = \frac{x}{2}$$

In general, a function g is the inverse of a function f if

$$g(f(x)) = x \text{ for all } x \text{ in the domain of } f$$

$$\text{and } f(g(x)) = x \text{ for all } x \text{ in the domain of } g$$

Example

Suppose we have a function f defined by

$$f(x) = 2x - 5$$

and we wish to find its inverse $f^{-1}(x)$. First, replace $f(x)$ with another letter. We will use y:

$$y = 2x - 5$$

Solve for x in terms of y:

$$y = 2x - 5$$

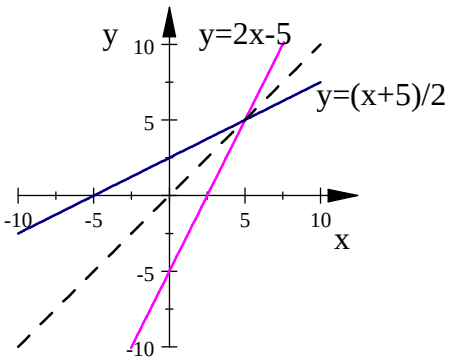
$$y + 5 = 2x$$

$$\frac{y + 5}{2} = x$$

This function has x as a function of y, that is, whenever y is input, the output is $\frac{y + 5}{2}$. This function is the inverse of function f, written f^{-1}. We will change the letter of the input variable to x and write

$$f^{-1}(x) = \frac{x + 5}{2}$$

Note that the graph of f and f^{-1} are symmetric about the line $y = x$:



If $f^{-1}$is the inverse function of f then the composition of f with f^{-1} is the identity function:

$$f \circ f^{-1}(x) = f(f^{-1}(x)) = x$$

and

$$f^{-1} \circ f(x) = f^{-1}(f(x)) = x$$

Example Using $f(x) = 2x - 5$ and $f^{-1}(x) = \frac{x+5}{2}$ we have

$$\begin{aligned} f(f^{-1}(x)) &= f\left(\frac{x+5}{2}\right) \\ &= 2\left(\frac{x+5}{2}\right) - 5 \\ &= x + 5 - 5 \\ &= x. \end{aligned}$$

It is also the case in this example that $f^{-1}(f(x)) = x$, and you should check this.

Here's an interactive Desmos app (https://www.desmos.com) which illustrates the graph of

$$f(x) = x^3 + 2$$

and its inverse

$$f^{-1}(x) = \sqrt[3]{x-2}$$

Use the slider to see the symmetry of points on the graph of f and f^{-1}. Note that the x- and y-coordinates of the points are exchanged.

Copy and paste this link into your browser:

https://www.desmos.com/calculator/hkffviunw2