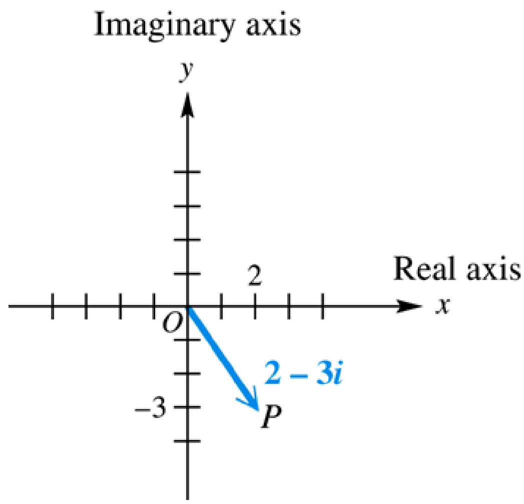
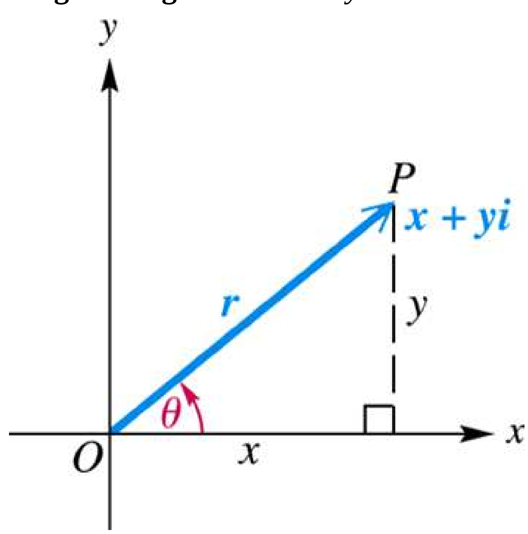
Complex Numbers in Rectangular and Polar Form

To represent complex numbers $x + yi$ geometrically, we use the rectangular coordinate system with the horizontal axis representing the real part and the vertical axis representing the imaginary part of the complex number.



We sketch a vector with initial point $(0,0)$ and terminal point $P(x,y)$. The length r of the vector is the **absolute value** or **modulus** of the complex number and the angle θ with the positive x-axis is the is called the **direction angle** or **argument** of $x + yi$.



Conversions between rectangular and polar form follows the same rules as it does for vectors.

Rectangular to Polar

For a complex number $x + yi$

$$|x + yi| = r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x},\ x \neq 0$$

Polar to Rectangular

$$x = r\cos\theta$$

$$y = r\sin\theta$$

The polar form $r(\cos\theta + i\sin\theta)$ is sometimes abbreviated

$$r\,\text{cis}\,\theta$$

Example

Convert $-\sqrt{3} + i$ to polar form.

Solution

$x = -\sqrt{3}$ and $y = 1$ so that

$$r = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = 2$$

and

$$\tan\theta = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Here the reference angle and for θ is $30°$. Since the complex number is in QII, we have

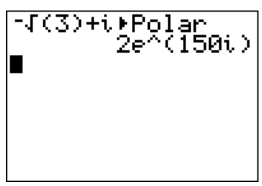
$$\theta = 180° - 30°$$

$$\theta = 150°$$

So that $-\sqrt{3} + i = 2\,\text{cis}\,150°$. In radian mode, we have

$$-\sqrt{3} + i = 2\,\text{cis}\,\frac{5\pi}{6}$$

Here's what the conversion looks like using the **Math/Complex** menu on the TI-83/84 (degree mode)



Example

In the case that $x = 0$ or $y = 0$, the conversions to polar form lead to quadrant angles.

$$-8i = 8\,\text{cis}\,270°$$

$$-5 = 5\,\text{cis}\,180°$$

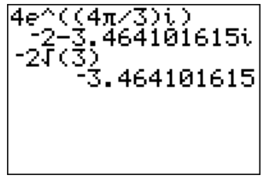
Example

Converting polar to rectangular form is straightforward.

$$\begin{aligned} 4\,\text{cis}\,240° &= 4\cos 240° + i\sin 240° \\ &= 4\left(-\frac{1}{2}\right) + i\left(-\frac{\sqrt{3}}{2}\right) \\ &= -2 - 2i\sqrt{3} \end{aligned}$$

Note that the i follows an integer or fraction but precedes a radical, an "unwritten rule" of mathematical writing style.

Warning: doing this conversion on the calculator requires **radian mode argument** and the radicals, of course, give decimal numbers.



Product and Quotient Theorems

The advantage of polar form is that multiplication and division are easier to accomplish.

Product Theorem

$$(r_1 \operatorname{cis}\theta_1)(r_2 \operatorname{cis}\theta_2) = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$$

Quotient Theorem

$$\frac{(r_1 \operatorname{cis}\theta_1)}{(r_2 \operatorname{cis}\theta_2)} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

Notice that the angle θ behaves in a manner analogous to that of the logarithms of products and quotients.

Example

Find $(2 \operatorname{cis} 45°)(3 \operatorname{cis} 135°)$ and convert the answer to rectangular form.

Solution

$$\begin{aligned}(2 \operatorname{cis} 45°)(3 \operatorname{cis} 135°) &= 2 \cdot 3 \operatorname{cis}(45° + 135°) \\ &= 6 \operatorname{cis} 180°\end{aligned}$$

In rectangular form, this answer is –6.

Example

Find $\frac{10 \operatorname{cis}(-60°)}{5 \operatorname{cis}(150°)}$ and convert the answer to rectangular form.

Solution

$$\begin{aligned}\frac{10 \operatorname{cis}(-60°)}{5 \operatorname{cis}(150°)} &= \frac{10}{5} \operatorname{cis}(-60° - 150°) \\ &= 2 \operatorname{cis}(-210°)\end{aligned}$$

Converting the polar result gives

$$\begin{aligned}2 \operatorname{cis}(-210°) &= 2(\cos(-210°) + i \sin(-210°)) \\ &= 2(\cos(210°) - i \sin(210°)) \\ &= 2\left(-\frac{\sqrt{3}}{2} - i\left(-\frac{1}{2}\right)\right) \\ &= -\sqrt{3} + i\end{aligned}$$

The advantage of using polar form will become even more pronounced when we calculate powers and roots of complex numbers using DeMoivre's Theorem.