Using the Factor Theorem and Rational Zeros Theorem

To solve higher degree polynomials without using the cubic or quartic formulas, we have two approaches:

1. The Factor Theorem

2. Approximate solution

Here are some examples of using the Factor Theorem

Example *Find all zeros of* $P(x) = 6x^3 - 29x^2 + 20x + 28$.

Solution: From inspection of the graph [you should set it up on your calculator] we see that $x = 2$ is a zero of $P(x)$. Set up synthetic division for the factor $(x - 2)$

$$\begin{array}{r|rrrr} 2 & 6 & -29 & 20 & 28 \\ & & 12 & -34 & -28 \\ \hline & 6 & -17 & -14 & \boxed{0} \end{array}$$

This shows that

$$P(x) = (x-2)(6x^2 - 17x - 14)$$

To find the other two zeros, solve the quadratic $6x^2 - 17x - 14$. Factoring gives

$$6x^2 - 17x - 14 = (3x+2)(2x-7)$$

and we have

$$S.S. = \left\{2, -\frac{2}{3}, \frac{7}{2}\right\}$$

Example *Find all zeros of* $P(x) = x^4 - 6x^3 + 10x^2 - 8$.

Solution: Close inspection of the graph shows that $x = 2$ is a possible *double* zero of $P(x)$. Set up two synthetic divisions for the factor $(x - 2)$.

$$\begin{array}{r|rrrrr} 2 & 1 & -6 & 10 & 0 & -8 \\ & & 2 & -8 & 4 & 8 \\ \hline & 1 & -4 & 2 & 4 & \boxed{0} \end{array}$$

so that $P(x) = (x-2)(x^3 - 4x^2 + 2x + 4)$. Then divide by $(x - 2)$ again:

$$\begin{array}{r|rrrr} 2 & 1 & -4 & 2 & 4 \\ & & 2 & -4 & -4 \\ \hline & 1 & -2 & -2 & \boxed{0} \end{array}$$

From this we have $P(x) = (x-2)(x-2)(x^2 - 2x - 2)$. The last factor can be solved by completing the square:

$$x^2 - 2x - 2 = 0$$
$$x^2 - 2x = 2$$
$$x^2 - 2x + 1 = 3$$
$$(x-1)^2 = 3$$
$$(x-1) = \pm\sqrt{3}$$
$$x = 1 \pm \sqrt{3}$$

and then the solution set of the quartic is

$$S.S. = \left\{2, 1+\sqrt{3}, 1-\sqrt{3}\right\}$$

where $x = 2$ is a double root of $P(x)$.

Example *Find a cubic polynomial with zeros* $x = 2$ *(multiplicity 2) and* $x = -5$*, and* y*-intercept* $(0,5)$*. Write the answer in factored form.*

Solution: Use the Factor Theorem to set up

$$P(x) = a_3(x-2)(x-2)(x+5)$$

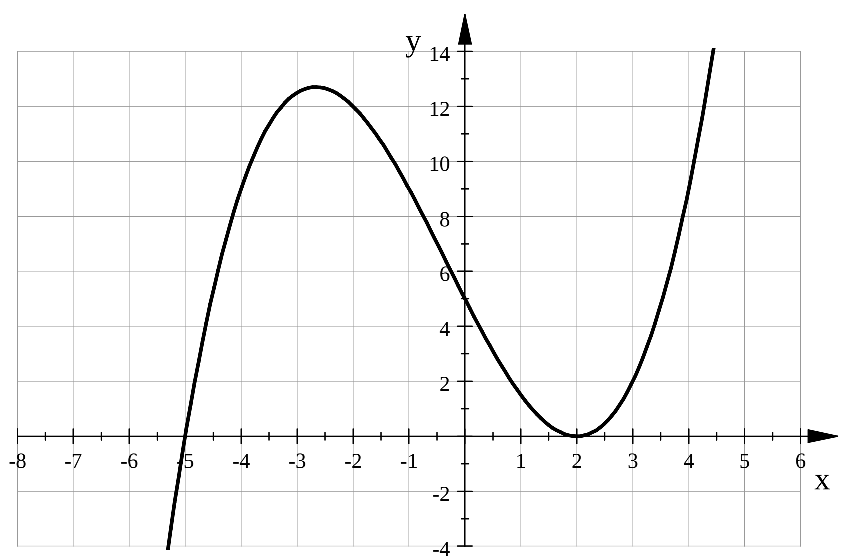
It remains to find the leading coefficient a_3. Since the y-intercept is $(0,5)$, we have

$$5 = a_3(0-2)(0-2)(0+5)$$
$$5 = 20a_3$$
$$\frac{1}{4} = a_3$$

Then we can write the polynomial

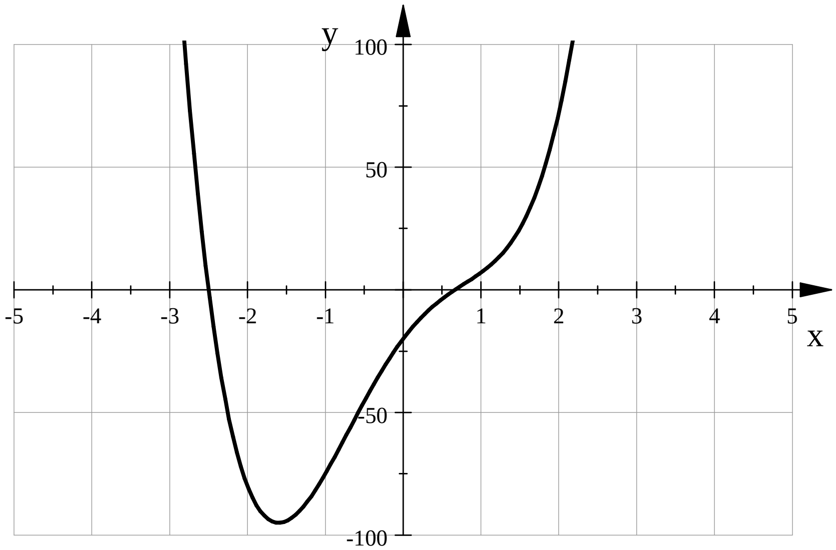
$$P(x) = \frac{1}{4}(x-2)(x-2)(x+5)$$

Here's the graph, which supports this result:



Example *Find all zeros of* $P(x) = 6x^4 - x^3 - 20x^2 + 42x - 20.$

Solution: First, examine the graph of the $P(x)$ in a window that will show all turning points and x –intercepts.



This graph is complete and shows that the polynomial has two real roots and two complex roots. However, neither of the real roots is an integer, and we must use the **Rational Zeros Theorem** to help identify a solution. The possible rational roots are fractions $\frac{b}{c}$ whose numerator b is a factor of -20 and denominator c is a factor of 6.

$$\frac{b}{c} = \pm\frac{1,2,5,10,20}{1,2,3,6} = \pm\left\{1,2,5,10,20,\frac{1}{2},\frac{5}{2},\frac{1}{3},\frac{2}{3},\frac{5}{3},\frac{10}{3},\frac{20}{3},\frac{1}{6},\frac{5}{6}\right\}$$

From inspecting the graph, we see that one root lies in the interval $(-3,-2)$, and the other is in the interval $(0,1)$. The candidates that meet these requirements are

$$-\frac{5}{2},\frac{1}{2},\frac{1}{3},\frac{2}{3},\frac{1}{6},\frac{5}{6}$$

Since $-\frac{5}{2}$ is the only negative candidate, we test it first with synthetic division

$$\begin{array}{r|rrrrr} -\frac{5}{2} & 6 & -1 & -20 & 42 & 20 \\ & & -15 & 40 & -50 & -20 \\ \hline & 6 & -16 & 20 & -8 & \boxed{0} \end{array}$$

This proves that $-\frac{5}{2}$ is a zero of $P(x)$ and that

$$\begin{aligned} P(x) &= \left(x+\frac{5}{2}\right)(6x^3 - 16x^2 + 20x - 8) \\ &= 2\left(x+\frac{5}{2}\right)(3x^3 - 8x^2 + 10x - 4) \\ &= (2x+5)(3x^3 - 8x^2 + 10x - 4) \end{aligned}$$

Since the other zeros of $P(x)$ must be zeros of $3x^3 - 8x^2 + 10x - 4$, the rational candidates to be checked must have denominator 3:

$$\frac{1}{3}, \frac{2}{3}$$

Inspection of the graph favors $\frac{2}{3}$ and synthetic division confirms it:

$$\begin{array}{c|cccc} \frac{2}{3} & 3 & -8 & 10 & -4 \\ & & 2 & -4 & 4 \\ \hline & 3 & -6 & 6 & \boxed{0} \end{array}$$

The polynomial

$$\begin{aligned} P(x) &= (2x+5)(3x^3 - 8x^2 + 10x - 4) \\ &= (2x+5)\left(x - \frac{2}{3}\right)(3x^2 - 6x + 6) \\ &= 3(2x+5)\left(x - \frac{2}{3}\right)(x^2 - 2x + 2) \\ &= (2x+5)(3x-2)(x^2 - 2x + 2) \end{aligned}$$

To find the remaining two zeros, solve $x^2 - 2x + 2 = 0$ to obtain $1 \pm i$ [you should check this step]. The solution set is

$$S.S. = \left\{-\frac{5}{2}, \frac{2}{3}, 1+i, 1-i\right\}$$

Closing Comment

What if the Rational Zeros Theorem fails to produce an exact zero of a polynomial? Then a calculator may be used to **approximate** the real solution(s) to a specified number of decimal places. In the case of cubic or quartic polynomials, there are formulas for the zeros (see Wikipedia), but they are tedious to implement, even when the coefficients are small integers.