## Solving Quadratic Equations

$$ax^2 + bx + c = 0, \quad a \neq 0$$

**Methods for solving**:

1. By **factoring**.
   - **A.** First, put the equation in standard form.
   - **B.** Then factor the left side
   - **C.** Set each factor = 0
   - **D.** Solve each equation

2. By **square root method**.
   - **A.** The solution set of

     $$x^2 = k$$

     is

     $$S.S. = \left\{ \sqrt{k}, -\sqrt{k} \right\}$$

   - **B.** If the left side of the equation is not a perfect square, then *complete the square* using the formula

     $$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

3. By **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

## The *Discriminant of the Quadratic Equation*

$$ax^2 + bx + c = 0, \quad a \neq 0$$

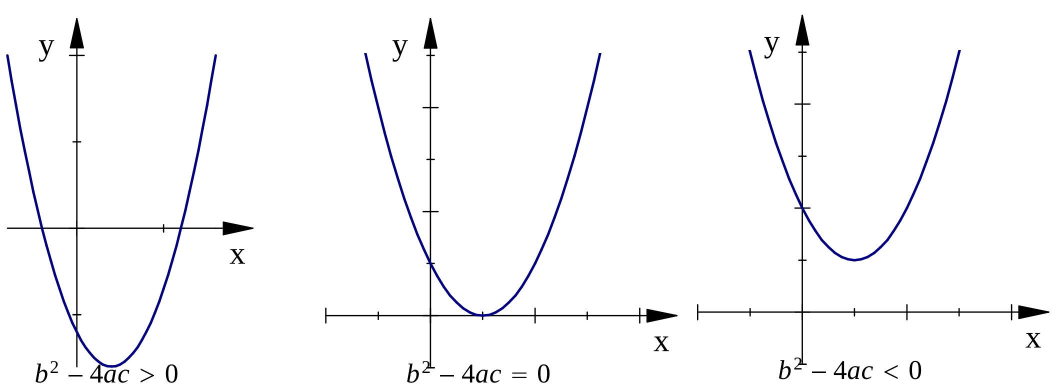
is the number

$$b^2 - 4ac$$

. This number tells us about the solutions of the equation. If

1. $b^2 - 4ac > 0$, there are two real solutions
2. $b^2 - 4ac = 0$, there is one double solution
3. $b^2 - 4ac < 0$, there are two complex non-real solutions
4. $a, b, c$ are integers and $b^2 - 4ac$ is a perfect square, the equation can be solved by factoring

Here's a sample of what the graphs of $y = ax^2 + bx + c$ when $a > 0$ look like:




## Examples

**Solving Quadratic Equation by Factoring**

Solve $x^2 + x = 12$

*Solution:*

$$x^2 + x - 12 = 0 \text{ place the equation in standard form}$$
$$(x+4)(x-3) = 0 \text{ factor the left side}$$
$$x + 4 = 0 \text{ or } x - 3 = 0 \text{ set each factor equal to 0}$$
$$x = -4 \text{ or } x = 3 \text{ and solve for } x$$
$$S.S. = \{-4, 3\} \text{ write the solution set}$$

**Solving Quadratic Equations by Square Root Method**

**1.** Solve $x^2 - 8 = 0$

*Solution:*

$$x^2 = 8 \text{ move the constant to the right side}$$
$$x = \pm\sqrt{8} \text{ take square root of both sides}$$
$$x = \pm 2\sqrt{2} \text{ and simplify}$$
$$S.S. = \left\{\pm 2\sqrt{2}\right\} \text{ write the solution set}$$

**2.** Solve $x^2 - 6x + 7 = 0$

*Solution:*

$$x^2 - 6x + 7 = 0$$
$$x^2 - 6x = -7 \text{ move constant}$$
$$x^2 - 6x + 9 = -7 + 9 \text{ complete square and balance equation}$$
$$(x-3)^2 = 2 \text{ factor and simplify}$$
$$x - 3 = \pm\sqrt{2} \text{ square root of both sides}$$
$$x = 3 \pm \sqrt{2} \text{ solve for } x$$
$$S.S. = \left\{3 \pm \sqrt{2}\right\} \text{ and write the solution set}$$

**3.** Solve $2x^2 - 5x - 6 = 0$

*Solution:*

$$2x^2 - 5x - 3 = 0$$
$$2x^2 - 5x = 3 \text{ move constant}$$
$$2\left(x^2 - \frac{5}{2}x\right) = 3 \text{ factor left side}$$
$$2\left(x^2 - \frac{5}{2}x + \frac{25}{16}\right) = 3 + 2 \cdot \frac{25}{16} \text{ complete square and balance}$$
$$2\left(x - \frac{5}{4}\right)^2 = \frac{49}{8} \text{ factor and simplify}$$
$$\left(x - \frac{5}{4}\right)^2 = \frac{49}{16} \text{ divide by 2}$$
$$x - \frac{5}{4} = \pm\frac{7}{4} \text{ square root of both sides}$$
$$x = \frac{5}{4} \pm \frac{7}{4} \text{ isolate } x$$
$$x = 3 \text{ or } -\frac{1}{2} \text{ simplify}$$
$$S.S. = \left\{3, -\frac{1}{2}\right\} \text{ write solution set}$$

**Solving Quadratic Equation by the Quadratic Formula**

**1.** Find the solutions of the quadratic equation

$$2x^2 - 4x - 1 = 0$$

*Solution* Use the quadratic formula and substitute $a = 2$, $b = 4$, $c = -1$.

$$
\begin{aligned}
x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2} \\
&= \frac{4 \pm \sqrt{16 + 8}}{4} \\
&= \frac{4 \pm \sqrt{24}}{4} \quad \text{the discriminant is 24} \\
&= \frac{4}{4} \pm \frac{2\sqrt{6}}{4} \quad \text{simplify radical} \\
&= 1 \pm \frac{\sqrt{6}}{2}
\end{aligned}
$$

The solution set is

$$S.S. = \left\{ 1 \pm \frac{\sqrt{6}}{2} \right\}$$

**2**. Find the zeros of the quadratic function

$$x^2 + 2x + 1$$

*Solution* By the quadratic formula, with $a = 1$, $b = 2$, $c = 1$

$$
\begin{aligned}
x &= \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} \\
&= \frac{-2 \pm \sqrt{0}}{2} \quad \text{the discriminant is 0} \\
&= -1
\end{aligned}
$$

The solution set is

$$S.S. = \{-1\}$$

For this quadratic there is only one zero; it is called a *double zero* or *double root*.

**3**. Find the zeros of $x^2 - 4x + 13$.

*Solution* . By the quadratic formula, with $a = 1$, $b = -4$, $c = 13$

$$
\begin{aligned}
x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 13}}{2 \cdot 1} \\
&= \frac{4 \pm \sqrt{16 - 52}}{2} \\
&= \frac{4 \pm \sqrt{-36}}{2} \quad \text{the discriminant is } -36
\end{aligned}
$$

Note that $\sqrt{-36} = 6i$ is not a real number. Thus, for this quadratic function, there are **no** real zeros. To complete the solution we must use **complex numbers**.

$$
\begin{aligned}
x &= \frac{4 \pm 6i}{2} \\
x &= 2 \pm 3i
\end{aligned}
$$

The solution set is

$$S.S. = \{2 + 3i, 2 - 3i\}$$

In many application problems, an approximate answer is needed.

**4**. **Example**: Approximate the solutions to three decimal places:

$$0.62x^2 - 4.31x + 2.94 = 0$$

**Solution**: Set up the quadratic formula

$$x = \frac{-(-4.31) \pm \sqrt{(-4.31)^2 - 4 \cdot 0.62 \cdot 2.94}}{2 \cdot 0.62}$$

on the calculator home screen to obtain $x = 6.184919776$ or $x = 0.766693128$.

Here's what one input line the home screen setup looks like:

```
(4.31+√(4.31²−4*.62*2.94))/(2*.62)
```

Round these off and write the solution set

$$S.S. = \{6.185, 0.767\}$$

This can also be solved on the graphing screen with $\mathbf{Y_1}$ as the quadratic function. Use the **[2nd]Calc/ 2.Zero** command.

## Solving Quadratic Inequalities

$$ax^2 + bx + c < 0 \text{ or}$$
$$ax^2 + bx + c \leq 0 \text{ or}$$
$$ax^2 + bx + c > 0 \text{ or}$$
$$ax^2 + bx + c \geq 0$$

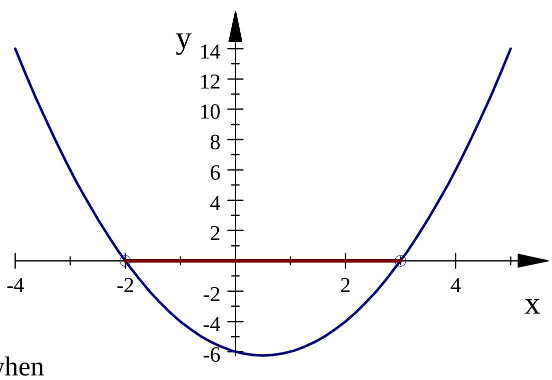
To find the solution set for these types of inequalities, find the *x*-intercepts of the graph of

$$y = ax^2 + bx + c$$

and note the interval on the *x*-axis where the graph is above or below the *x*-axis. The solution set will be the region *between* or *outside* the *x*-intercepts.

**Example**: Solve $x^2 - x - 6 < 0$.

Find the *x*-intercepts by solving $x^2 - x - 6 = 0$ to obtain $x = 3$ or $x = -2$. These are called *critical numbers* and are the *x*-intercepts of the graph of $y = x^2 - x - 6$. Now examine the graph:




$x^2 - x - 6$

The quadratic function $x^2 - x - 6$ is negative when

$$-2 < x < 3$$

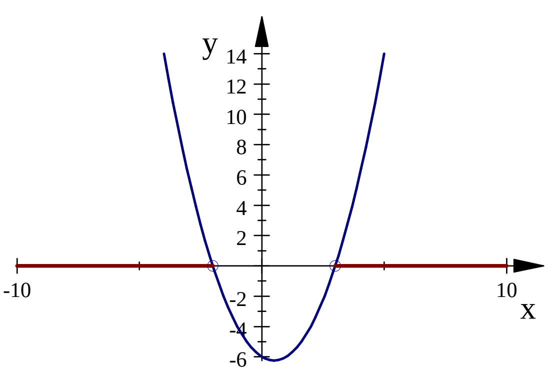
Using interval notation, the **Solution Set** is

$$S.S. = (-2, 3)$$

**Example**:To solve

$$x^2 - x - 6 \geq 0$$

examine the same graph to obtain




$x^2 - x - 6$

The quadratic function $x^2 - x - 6$ is nonnegative (greater than or equal to zero) when

$$x \leq -2 \text{ or } x \geq 3$$

Note the use of the word "**or**" in this description. The word "and" is incorrect because it indicates that both inequalities are true at the same time. A number cannot be less than or equal to $-2$ at the same time it is greater than or equal to 3. Here's the solution set in **interval notation**, using the union symbol.

$$S.S. = (-\infty, -2] \cup [3, \infty)$$

**Note**: The *test menu* on the TI calculators may also be used to obtain a good visual representation of the $S.S.$