Graphs of Rational Functions

A **rational function** is a function that can be written as the quotient of two polynomial functions, with the degree of the denominator at least 1. Let us write this as

$$y = \frac{P(x)}{Q(x)} = \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0}, a_m \neq 0, b_n \neq 0, n \geq 1$$

Here is how to find asymptotes.

I. **Vertical asymptotes**

Set $Q(x) = 0$ and solve for x. Each **real** solution r for which $Q(r) = 0$ **and** $P(r) \neq 0$ gives a vertical asymptote $x = r$.

What if there are no **real** zeros of $Q(x)$? Then there are no vertical asymptotes of the graph of y.

What if r is a real zero of both $Q(x)$ and $P(x)$? Then the graph of y has a **hole** at $x = r$, but no vertical asymptote at $x = r$.

II. **Horizontal asymptotes**

A. If $m < n$ then $y = 0$ (the x-axis) is the horizontal asymptote.

B. If $m = n$ then

$$y = \frac{a_m}{b_n}$$

(quotient of leading coefficients) is a horizontal asymptote.

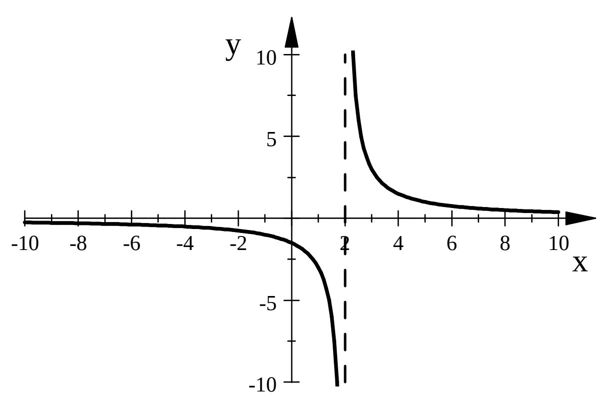
C. If $m > n$ then there is **no** horizontal asymptote.

III. **Oblique or Slant Asymptotes**

If $m = n + 1$ then there is an oblique asymptote which can be found by performing division of the polynomials. The quotient is a linear polynomial which defines the slant asymptote.

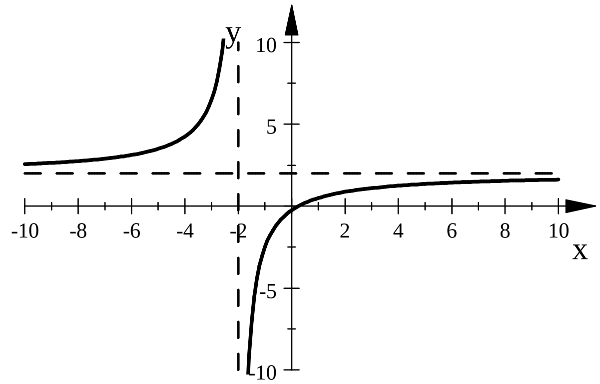
Here are some examples of rational functions:

Example 1. $y = \dfrac{3}{x-2}$



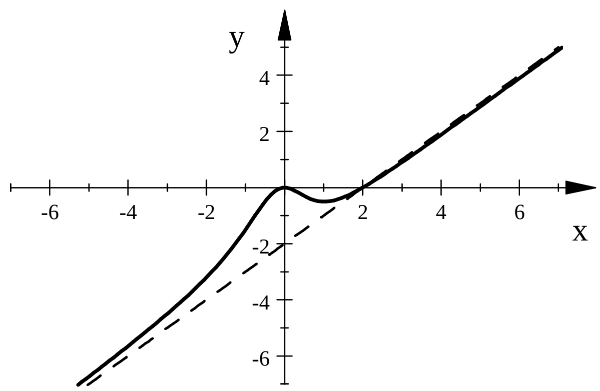
Note that the vertical line $x = 2$ is a vertical asymptote and that $y = 0$ (the x-axis) is a horizontal asymptote.

Example 2. $y = \dfrac{4x - 1}{2x + 4}$



The vertical asymptote is the line $x = -2$ and the horizontal asymptote is the line $y = 2$.

Example 3. $y = \dfrac{x^3 - 2x^2}{x^2 + 1}$



This function has no vertical asymptotes since the equation $x^2 + 1 = 0$ has no **real** solutions (its solutions are $x = i$ and $x = -i$). It has no horizontal asymptote since the degree of the numerator is greater than the degree of the denominator. But it does have a slant asymptote: by long division

$$\frac{x^3 - 2x^2}{x^2 + 1} = x - 2 + \frac{-x + 2}{x^2 + 1}$$

The oblique (or slant) asymptote is the line

$$y = x - 2$$

This example illustrates the fact that the graph of a rational function **may** cross a horizontal or slant asymptote in the "middle" but not at the ends of the graph.

Final Note: To enter these examples into a calculator, you must enclose the numerator and denominator in parentheses. For example 2 above, use

$$Y_1=(4X-1)/(2X+4)$$

Since the calculator is (usually) in connected mode, a zoom standard window may cause the graph to produce a false vertical line near $x = -2$ in older models (TI-83s or TI-82s). This artifact can be eliminated by graphing in dot mode. The newer models of the TI-84, or TI-84s with updated operating systems will graph rational functions correctly.