Essential Trigonometry Without Geometry

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Abstract

The development of the trigonometric functions in introductory texts usually follows geometric constructions using right triangles or the unit circle. While these methods are satisfactory at the elementary level, advanced mathematics demands a more rigorous approach. Our purpose here is to revisit elementary trigonometry from an entirely analytic perspective. We will give a comprehensive treatment of the sine and cosine functions and will show how to derive the familiar theorems of trigonometry without reference to geometric definitions or constructions.

Introduction

Our purpose in this paper is to show how the definitions and theorems of elementary trigonometry can be developed without references to geometric constructions. We will use methods from real analysis to provide an alternate view of the sine and cosine functions. Along the way we will see a relationship that leads to a non-geometric construction of pi. Finally, we will make connections with the familiar geometric approach. For this study, we will assume a familiarity with calculus, differential equations, and real analysis.

Definitions and Basic Properties

We begin by considering the solution of the second-order homogeneous linear differential equation

$$f''(x) + f(x) = 0 \text{ with } f(0) = 0 \text{ and } f'(0) = 1.$$

By the Existence and Uniqueness Theorem we know that a unique solution exists [Nagle, Saff, and Snider, p. 171]. If this solution has a power series representation around the ordinary point $x = 0$, it must have the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

Note that $f(0) = c_0 = 0$ and $f'(0) = c_1 = 1$. We also have

$$\begin{aligned} f''(x) &= \sum_{n=2}^{\infty} (n)(n-1)\, c_n x^{n-2} \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2} x^n \end{aligned}$$

Then

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2} x^n + \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \left((n+2)(n+1)\, c_{n+2} + c_n\right) x^n = 0$$

Since this power series is 0 for all x, we get the general recursion relation

$$(n+2)(n+1)\, c_{n+2} + c_n = 0$$

so that

$$c_{n+2} = -\frac{c_n}{(n+2)(n+1)}.$$

Because $c_0 = 0$, we have for all even indices $2n$

$$c_{2n} = 0$$

Let us now examine the coefficients with odd indices $2n+1$.

$$\begin{aligned} c_1 &= 1 \quad \text{initial condition} \\ c_3 &= -\frac{1}{3\cdot 2} = -\frac{1}{3!} \\ c_5 &= -\frac{-\frac{1}{3!}}{5\cdot 4} = \frac{1}{5!} \\ c_7 &= -\frac{\frac{1}{5!}}{7\cdot 6} = -\frac{1}{7!} \end{aligned}$$

and in general,

$$c_{2n+1} = (-1)^n \frac{1}{(2n+1)!}$$

The power series about $x = 0$ must have the form

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Using the Ratio Test, it is easy to show that this series converges for all real x.

The function represented by this power series is the unique solution of the differential equation

$$f''(x) + f(x) = 0 \text{ with } f(0) = 0 \text{ and } f'(0) = 1.$$

We call this function the **sine** function**,** denoted $\sin x$, or $\sin(x)$.

Definition *Sine Function*

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

We define the **cosine** to be the derivative of the sine function.

Definition *Cosine Function*

$$\cos x = \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

The following are elementary consequences of the definitions.

1. $\sin 0 = 0$
2. $\cos 0 = 1$
3. The function $\sin x$ is odd because all exponents in its power series are odd.
4. The function $\cos x$ is even because all exponents in its power series are even.

5. The functions $\sin x$ and $\cos x$ are both continuous since they are differentiable.

6. The derivatives of $\sin x$ are cyclic with order four.

$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f''''(x)$
$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$

Key Theorems

This section presents the Pythagorean and Sine Sum identities which, along with the smallest positive critical value of $\sin x$, enable the development of several important identities and analytic results in elementary trigonometry.

First, we prove the Pythagorean Identity.

Theorem *Pythagorean Identity* For all x,

$$\sin^2 x + \cos^2 x = 1$$

Proof: Consider the derivative of the left side.

$$\begin{aligned} \frac{d}{dx}\left(\sin^2 x + \cos^2 x\right) &= 2\sin x\cos x + 2\cos x\left(-\sin x\right) \\ &= 0 \end{aligned}$$

Since the derivative is 0, $\sin^2 x + \cos^2 x$ is a constant.

Because $\sin 0 = 0$, and $\cos 0 = 1$, this constant must be 1. □

Next, we consider the identity for the sine of the sum of x and y. The proof in most elementary trigonometry texts involves a geometric construction with triangles or the unit circle. In our geometry-free approach, we will use only power series.

Theorem *Sine Sum Identity* For all x, y,

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$

Proof: Consider the series expansion

$$\sin(x+y) = \sum_{n=0}^{\infty}(-1)^n\frac{(x+y)^{2n+1}}{(2n+1)!}$$

Now examine the general n^{th} term a_n of this series using the Binomial Theorem:

$$\begin{aligned}
a_n &= (-1)^n \frac{(x+y)^{2n+1}}{(2n+1)!} \\
&= \frac{(-1)^n}{(2n+1)!}(x+y)^{2n+1} \\
&= \frac{(-1)^n}{(2n+1)!}\sum_{i=0}^{2n+1}\binom{2n+1}{i}x^{2n+1-i}y^i \\
&= \frac{(-1)^n}{(2n+1)!}\sum_{i=0}^{2n+1}\frac{(2n+1)!}{i!\,(2n+1-i)!}x^{2n+1-i}y^i \\
&= (-1)^n \sum_{i=0}^{2n+1}\frac{1}{i!\,(2n+1-i)!}x^{2n+1-i}y^i
\end{aligned}$$

This last sum has $2n+2$ terms. We will re-write it as two sums each having $n+1$ terms.

$$\begin{aligned}
(-1)^n \sum_{i=0}^{2n+1}\frac{1}{i!\,(2n+1-i)!}x^{2n+1-i}y^i &= \underbrace{(-1)^n \sum_{i=0}^{n}\frac{x^{2i+1}y^{2n-2i}}{(2i+1)!\,(2n-2i)!}}_{\text{increasing odd powers of } x} + \underbrace{(-1)^n \sum_{i=0}^{n}\frac{x^{2n-2i}y^{2i+1}}{(2n-2i)!\,(2i+1)!}}_{\text{decreasing even powers of } x} \\
&= \frac{(-1)^n}{(2n+1)!}\sum_{i=0}^{n}\frac{(2n+1)!x^{2i+1}y^{2n-2i}}{(2i+1)!\,(2n-2i)!} + \frac{(-1)^n}{(2n+1)!}\sum_{i=0}^{n}\frac{(2n+1)!x^{2n-2i}y^{2i+1}}{(2n-2i)!\,(2i+1)!} \\
&= \underbrace{\frac{(-1)^n}{(2n+1)!}\sum_{i=0}^{n}\binom{2n+1}{2i+1}x^{2i+1}y^{2n-2i}}_{[1]} + \underbrace{\frac{(-1)^n}{(2n+1)!}\sum_{i=0}^{n}\binom{2n+1}{2i+1}x^{2n-2i}y^{2i+1}}_{[2]}
\end{aligned}$$

This last line represents the n^{th} term of the expansion of $\sin(x+y)$. We now turn our attention to the right side

$$\sin x \cos y + \cos x \sin y$$

and consider the series expansion of the term $\sin x \cos y$.

Since the series for $\sin x$ and for $\cos x$ both converge absolutely, we can write $\sin x \cos y$ as the Cauchy product of the two series

$$\sin x \cos y = \sum_{n=0}^{\infty} c_n$$

where

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}\,, \quad n = 0, 1, 2, 3, \ldots$$

and the a_i, b_{n-i} terms come from the series for $\sin x$ and $\cos x$, respectively [Rudin, p. 63ff]. Let us examine the general term c_n of this Cauchy product.

$$\begin{aligned}
c_n &= \sum_{i=0}^{n} (-1)^i \, \frac{x^{2i+1}}{(2i+1)!} \cdot (-1)^{n-i} \, \frac{y^{2n-2i}}{(2n-2i)!} \\
&= \sum_{i=0}^{n} (-1)^n \, \frac{x^{2i+1} y^{2n-2i}}{(2i+1)!\,(2n-2i)!} \\
&= \frac{(-1)^n}{(2n+1)!} \sum_{i=0}^{n} \frac{(2n+1)!}{(2i+1)!\,(2n-2i)!} x^{2i+1} y^{2n-2i} \\
&= \frac{(-1)^n}{(2n+1)!} \sum_{i=0}^{n} \binom{2n+1}{2n-2i} x^{2i+1} y^{2n-2i}
\end{aligned}$$

Then the term c_n is the odd powers of x in part [1] of the general binomial expansion above.

By switching x with y in the previous equation, we get the general term d_n for the Cauchy product of the series for $\sin y$ and $\cos x$.

$$\begin{aligned}
d_n &= \frac{(-1)^n}{(2n+1)!} \sum_{i=0}^{n} \binom{2n+1}{2n-2i} y^{2i+1} x^{2n-2i} \\
&= \frac{(-1)^n}{(2n+1)!} \sum_{i=0}^{n} \binom{2n+1}{2n-2i} x^{2n-2i} y^{2i+1} \\
&= \frac{(-1)^n}{(2n+1)!} \sum_{i=0}^{n} \binom{2n+1}{2i+1} x^{2n-2i} y^{2i+1}
\end{aligned}$$

This matches the even powers of x in part [2] of the general binomial expansion.

Therefore

$$a_n = c_n + d_n$$

and

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

□

We now turn our attention to a special value, the smallest positive critical value of $\sin x$, a number we will call Q.

Theorem *Critical Value* There exists a smallest positive critical value of $\sin x$, that is, a smallest positive zero of $\cos x$.

Proof. We have already seen that $\cos 0 = 1$. Now observe that

$$\cos 2 = 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} + \cdots$$

We now write

$$\begin{aligned}\cos 2 &= \left(1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!}\right) + R_3 \\ &= \left(-\frac{19}{45}\right) + R_3 \\ &\leq -\frac{19}{45} + |R_3|\end{aligned}$$

The Remainder Theorem for alternating series tells us that

$$\begin{aligned}|R_3| \leq a_4 = \frac{2^8}{8!} \quad \text{and so} \\ \cos 2 \leq -\frac{19}{45} + \frac{2}{315} = -\frac{131}{315}\end{aligned}$$

Since $\cos 0 > 0$ and $\cos 2 < 0$, by the Intermediate Value Theorem, there is at least one real number $c \in (0, 2)$ with $\cos c = 0$. The nonempty set $\{x | \cos x = 0\}$ is the inverse image of the closed point set $\{0\}$ under the continuous function $\cos x$. Therefore the set $\{x | \cos x = 0\}$ is closed. It follows that the set

$$\{x | \cos x = 0\} \cap [0, 2]$$

is nonempty, closed, bounded, and is therefore compact [Willard, p. 120]. It must contain its least element which we shall call, temporarily, Q.

□

Definition of Q

$$Q = \min\left(\{x | \cos x = 0\} \cap [0, 2]\right)$$

Consequences of the Key Theorems

The Pythagorean Identity leads directly to the following corollary.

Corollary For all x,

$$|\sin x| \leq 1 \quad \text{and} \quad |\cos x| \leq 1.$$

Proof: If $|\sin x| > 1$, then $\cos^2 x < 0$ and $\cos x$ is not a real number. Similarly, if $|\cos x| > 1$, then $\sin x$ is not a real number. In this study, we are restricting our work to real numbers.

□

The next two corollaries follow from the Pythagorean Identity and the special properties of Q.

Corollary $\sin Q = 1$ and $\sin x$ has an absolute maximum value of 1 at $x = Q$.

Proof: Since $\cos 0 = 1$ and $\cos x$ is an even function, for $x \in (-Q, Q)$, we have $\cos x > 0$. Therefore $\sin x$ is strictly increasing on $(-Q, Q)$. Since $0 < Q$ we have $0 = \sin 0 < \sin Q$. From the Pythagorean Identity we know that

$$\sin^2 Q + \cos^2 Q = 1$$

Since $\cos Q = 0$, it must be the case that $\sin Q = 1$. We have already observed that

$$|\sin x| \leq 1$$

and therefore 1 is an absolute maximum of $\sin x$. □

Corollary The range of $\sin x$ is $[-1, 1]$.

Proof: Because $\sin x$ is an odd function we have $\sin(-Q) = -\sin Q = -1$ is an absolute minimum. The range $[-1, 1]$ follows from the continuity of $\sin x$ and the Intermediate Value Theorem. □

Later we will see that the range of cosine is also $[-1, 1]$.

Our next two corollaries follow from the Sine Sum Theorem.

Corollary $\sin(x - y) = \sin x \cos y - \cos x \sin y$

Proof: Because $\sin x$ is an odd function and $\cos x$ is even, we have the following:

$$\begin{aligned} \sin(x - y) &= \sin(x + (-y)) \\ &= \sin x \cos(-y) + \cos x \sin(-y) \\ &= \sin x \cos y - \cos x \sin y \end{aligned}$$

□

Corollary $\sin 2x = 2 \sin x \cos x$

Proof:

$$\begin{aligned} \sin 2x &= \sin(x + x) \\ &= \sin x \cos x + \cos x \sin x \\ &= 2 \sin x \cos x \end{aligned}$$

□

We now consider the cofunction rules that follow from the Sine Sum Identity and the properties of Q. We will use these later to show that the sine and cosine functions are periodic.

Corollary *Cofunction Rule* $\sin(Q - x) = \cos x$

Proof:

$$\begin{aligned}
\sin(Q - x) &= \sin Q \cos x - \cos Q \sin x \\
&= 1 \cdot \cos x - 0 \cdot \sin x \\
&= \cos x
\end{aligned}$$

□

Corollary *Cofunction Rule* $\cos(Q - x) = \sin x$

Proof:

$$\begin{aligned}
\cos(Q - x) &= \sin(Q - (Q - x)) \\
&= \sin x
\end{aligned}$$

□

In the following corollaries we complete the sum, difference, and double angle rules.

Corollary $\cos(x + y) = \cos x \cos y - \sin x \sin y$

Proof:

$$\begin{aligned}
\cos(x + y) &= \sin(Q - (x + y)) \\
&= \sin((Q - x) - y) \\
&= \sin(Q - x)\cos y - \cos(Q - x)\sin y \\
&= \cos x \cos y - \sin x \sin y
\end{aligned}$$

□

The following corollaries now follow.

Corollary $\cos(x - y) = \cos x \cos y + \sin x \sin y$

Proof:

$$\begin{aligned}\cos(x-y) &= \cos(x+(-y)) \\ &= \cos x\cos(-y) - \sin x\sin(-y) \\ &= \cos x\cos y + \sin x\sin y\end{aligned}$$

□

Corollary $\cos 2x = 2\cos^2 x - 1$

Proof:

$$\begin{aligned}\cos 2x &= \cos(x+x) \\ &= \cos x\cos x - \sin x\sin x \\ &= \cos^2 x - \sin^2 x \\ &= \cos^2 x - (1-\cos^2 x) \\ &= 2\cos^2 x - 1\end{aligned}$$

□

We have seen that the three key theorems have led to the familiar difference formulas as well as double angle formulas. From these follow the other identities such as half-angle and product-to-sum rules. In particular, we will later need the identity

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

Periodicity

We will need the sine and cosine function values of $4Q$ to show periodicity. Here is a sequence of steps to arrive at this point.

1. $\sin 2Q = 2\sin Q\cos Q = 2(1)(0) = 0$
2. $\cos 2Q = \sin(Q-2Q) = \sin(-Q) = -\sin Q = -1$.

   From this it follows that the range of $\cos x$ is $[-1,1]$.
3. $\sin 3Q = \sin(Q+2Q) = \sin Q\cos 2Q + \cos Q\sin 2Q = -1$
4. $\cos 3Q = \sin(Q-3Q) = \sin(-2Q) = -\sin 2Q = 0$

5. $\sin 4Q = 2\sin 2Q \cos 2Q = 0$

6. $\cos 4Q = \sin(Q - 4Q) = \sin(-3Q) = -\sin(3Q) = -(-1) = 1$

We now have the machinery needed to prove the periodicity of $\sin x$ and $\cos x$.

Definition A function $f(x)$ is *periodic* if there is a positive number p such that

$$f(x+p) = f(x)$$

for all x. If there is a *smallest* positive number p for which this holds, then p is called the *period of* f.

Theorem *Periodicity of Sine* The sine function is periodic and its period is $4Q$.

Proof: We first show that sine is periodic.

$$\begin{aligned}\sin(x+4Q) &= \sin x \cos 4Q + \cos x \sin 4Q \\ &= \sin x(1) + \cos x(0) \\ &= \sin x\end{aligned}$$

This shows that $\sin x$ is periodic, but does not show that the period is $4Q$. To show that $4Q$ is the period, assume, to the contrary, that there exists a number R such that $0 < 4R < 4Q$ and for all x,

$$\sin(x+4R) = \sin x$$

Observe that $0 < R < Q$. For $x \in (0, Q)$ we have $\cos x > 0$ because $\cos 0 = 1$ and Q is the smallest value with $\cos Q = 0$. We also have $\sin x > 0$ since $\sin 0 = 0$ and sin is increasing on $(0, Q)$. Now examine $\sin Q$:

$$\begin{aligned}\sin Q &= \sin(Q+4R) \\ &= \sin Q \cos 4R + \cos Q \sin 4R \\ &= \cos 4R \\ &= \cos 2(2R) \\ &= 2\cos^2(2R) - 1\end{aligned}$$

Because $\sin Q = 1$,

$$\begin{aligned}1 &= 2\cos^2(2R) - 1 \\ 1 &= \cos^2(2R) \\ \cos 2R = 1 &\text{ or } \cos 2R = -1\end{aligned}$$

We now have two cases:

Case I: $\cos 2R = 1$.

Then by the double angle identity,

$$2\cos^2 R - 1 = 1$$
$$\cos^2 R = 1$$

If $\cos^2 R = 1$, then by the Pythagorean Identity, $\sin R = 0$, a contradiction to the fact that $\sin R > 0$.

Case II: $\cos 2R = -1$.

Then

$$2\cos^2 R - 1 = -1$$
$$\cos R = 0$$

This last statement contradicts the choice of Q as the smallest positive number in $[0, 2]$ with $\cos Q = 0$. Therefore such a number R does not exist, and the period of sin is $4Q$. □

Corollary *Periodicity of Cosine* The cosine function is periodic with period $4Q$.

Proof: We can write $\cos x$ as

$$\cos x = -\sin(x - Q)$$

Because horizontal translations and vertical rotations about the x-axis do not change the period of a function, $\cos x$ is periodic with period $4Q$. □

Connection to Geometry

With this result we now show the connection between the analytic and geometric approaches to trigonometry.

Theorem *Connection with* π

$$\int_0^1 \sqrt{1 - x^2}\, dx = \frac{Q}{2}$$

Proof: Use the substitution

$$x = \sin\theta$$

with the values

x	θ
0	0
1	Q

so that the integral becomes

$$
\begin{aligned}
\int_0^Q \sqrt{1-\sin^2\theta}\cos\theta\,d\theta &= \int_0^Q \cos^2\theta\,d\theta \\
&= \int_0^Q \frac{1}{2}\left(1+\cos 2\theta\right)\,d\theta \\
&= \frac{1}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^Q \\
&= \frac{1}{2}\left[\left(Q+\frac{1}{2}\sin 2Q\right)-\left(0+\frac{1}{2}\sin\left(2\cdot 0\right)\right)\right] \\
&= \frac{1}{2}Q
\end{aligned}
$$

□

The integral $\int_0^1 \sqrt{1-x^2}\,dx$ represents the quarter-circle area enclosed by the unit circle, the nonnegative x-axis, and the nonnegative y-axis, and so we are led to the conclusion that

$$Q = \pi/2$$

.

Using what we have previously developed about multiples of Q, we have a table restating the values for sine and cosine in terms of π instead of Q.

x	0	$\pi/2$	π	$3\pi/2$	2π
$\sin x$	0	1	0	-1	0
$\cos x$	1	0	-1	0	1

From this follows the usual information about the graphs of the sine and cosine: intervals for positive/negative values, intervals for increasing/decreasing, local (and absolute) maximums/minimums.

Without geometry, we can find the values of sine and cosine of $\frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{3}$ using only the sum and difference identities. We include the development of these values in Appendix A. In Appendix B we present the mathematics that connects the sine and cosine functions, defined here as power series, to the trig functions defined using the unit circle.

Pythagorean Identity Revisited

We conclude this study with the observation that the **converse** of the Pythagorean Identity also holds.

Theorem If $f:\mathbb{R}\to\mathbb{R}$ is analytic, $f'(0)=1$, $f(0)=0$, and f satisfies the Pythagorean Identity

$$\left(f(x)\right)^2+\left(f'(x)\right)^2=1$$

for all x, then $f(x)\equiv\sin x$.

Proof: Differentiation of both sides gives

$$2\left(f\left(x\right)\right)f'\left(x\right)+2f'\left(x\right)f''\left(x\right)=0$$

so that

$$2f'\left(x\right)\left(f\left(x\right)+f''\left(x\right)\right)=0$$

Since $f'(0)=1$, and f is analytic, f' is positive on some open interval containing 0. Therefore, **on this interval**,

$$f\left(x\right)+f''\left(x\right)=0,$$

and $f(x)=\sin(x)$. Moreover, if two analytic functions agree on an open interval, then they agree on $\mathbb{R}$.

$\square$

Summary and Conclusions

We have developed the theorems and identities of basic trigonometry using the definition of the sine function as the solution, expressed as a power series, of a certain second order linear homogeneous differential equation. The key theorems in this study are the Pythagorean Identity, the Sine Sum Identity, and the special value Q, which turned out to be $\pi/2$. From these the other identities follow. The interested reader is referred to Landau, chapter 16, in which the sine and cosine functions are developed from a power series definition. In a brief note, Appendix III in Hardy uses the definition of the inverse tangent function as an integral to lead to the definitions of sine, cosine, and their sum laws.

In a future study we plan to consider a generalization of the sine and cosine functions, and show that versions of the Key Theorems still hold in these settings.

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Appendix A: Trig Functions of Special Angles

First we consider $\frac{\pi}{4}$.

$$\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$$

Since

$$1 = \sin^2\left(\frac{\pi}{4}\right) + \cos^2\left(\frac{\pi}{4}\right) = 2\sin^2\left(\frac{\pi}{4}\right)$$

we obtain

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \cos\left(\frac{\pi}{4}\right)$$

To find values for $\sin\frac{\pi}{6}$, we need the triple-angle identity

$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$$

This follows from expanding $\sin(2\theta+\theta)$. We can now write

$$\begin{aligned} 1 &= \sin\frac{\pi}{2} \\ &= \sin\left(3\cdot\frac{\pi}{6}\right) \\ &= 3\sin\frac{\pi}{6} - 4\sin^3\frac{\pi}{6} \end{aligned}$$

We solve this cubic equation in $\sin\frac{\pi}{6}$ to obtain a double solution $\frac{1}{2}$ and single solution -1. Because $\sin\frac{\pi}{6} > 0$,

$$\begin{aligned} \sin\frac{\pi}{6} &= \frac{1}{2}, \text{ and} \\ \cos\frac{\pi}{6} &= \frac{\sqrt{3}}{2} \end{aligned}$$

Here's a summary table:

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin x$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos x$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0

Appendix B: Connection to Unit Circle Trigonometry

Lemma If f is continuous on $[a,b]$ and strictly increasing on (a,b), then f is strictly increasing on $[a,b]$.

Proof: We first show that for any x in the interval (a,b), we must have $f(a) < f(x)$. Assume, to the contrary, that there exists c, $a < c < b$, such that $f(a) \geq f(c)$.

Case I: $f(a) > f(c)$. Let

$$\varepsilon = \frac{f(c) - f(a)}{2}$$

Then by right-hand continuity of f at a, there exists δ, $0 < \delta < c - a$, such that if $a < x < a + \delta$ then

$$f(a) - \varepsilon < f(x) < f(a) + \varepsilon$$

Then

$$\begin{aligned} f(x) &> f(a) - \frac{f(c) - f(a)}{2} \\ &= \frac{f(a) + f(c)}{2} \\ &> \frac{f(c) + f(c)}{2} \\ &= f(c) \end{aligned}$$

Since x, c are both in (a, b) and $x < a + \delta < a + (c - a) = c$, we must have $f(x) < f(c)$, a contradiction. Therefore it cannot be the case that $f(a) \geq f(c)$ for some $c \in (a, b)$.

Case II: $f(a) = f(c)$. Then consider $\frac{a+c}{2} \in (a, c) \subset (a, b)$. Since f is strictly increasing on (a, b), it follows that

$$f\left(\frac{a + c}{2}\right) < f(c) = f(a)$$

and we have the situation of Case I.

A similar argument shows that if $a < x < b$, then $f(x) < f(b)$.

□

Theorem The function $\sin x$ is strictly increasing on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Proof: By our development and definition of $\frac{\pi}{2}$ as least in $[0, 2]$ with $\cos\left(\frac{\pi}{2}\right) = 0$, $\cos x$ is positive on $\left[0, \frac{\pi}{2}\right)$. Since it is an even function, it is positive on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and therefore $\sin x$ is increasing on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The function $\sin x$ is differentiable and therefore continuous at all x, and by the lemma must be increasing on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

□

We see, then, that the sine function restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is a bijection onto $[-1, 1]$.

Definition *Inverse Sine Function* The *inverse sine function of x*, written here as $\arcsin x$, is defined:

$$\arcsin x : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

is the inverse of the sine function restricted to the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

We now consider the derivative of $\arcsin x$, $-1 < x < 1$.

Theorem If $-1 < x < 1$ then $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$.

Proof: If $y = \arcsin x$, $-1 < x < 1$, then $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and

$$\sin y = x$$

Using implicit differentiation,

$$\cos y \frac{dy}{dx} = 1$$

and

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

What is $\cos y = \cos(\arcsin x)$? By the Pythagorean Identity,

$$\cos^2 y + \sin^2 y = 1$$

$$\cos^2 y + x^2 = 1$$

and so $\cos y = \sqrt{1-x^2}$ or $\cos y = -\sqrt{1-x^2}$. Since $-\frac{\pi}{2} < y < \frac{\pi}{2}$, we have $\cos y > 0$. Thus $\cos y = \sqrt{1-x^2}$ and

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

□

Corollary If $-1 < x < 1$, then

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-t^2}} dt$$

Proof: This follows from the Fundamental Theorem of Calculus.

□

Does this equation hold for $x = 1$ and for $x = -1$?

Theorem

$$\int_0^1 \frac{1}{\sqrt{1-t^2}} dt = \frac{\pi}{2}$$

Proof: Let $g(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}}$, $0 \leq x \leq 1$.

What is the value of $g(1)$, an improper integral? First, we know that $g(x) = \arcsin x$ for $x \in [0, 1)$ and that

$$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

Therefore

$$g\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

We are now ready to find $g(1)$.

$$\begin{aligned}
g(1) &= \lim_{b\to 1^-}\int_0^b \frac{dt}{\sqrt{1-t^2}} \\
&= \int_0^{\sqrt{2}/2}\frac{dt}{\sqrt{1-t^2}} + \lim_{b\to 1^-}\int_{\sqrt{2}/2}^b \frac{dt}{\sqrt{1-t^2}} \\
&= g\left(\tfrac{\sqrt{2}}{2}\right) + \lim_{b\to 1^-}\int_{\sqrt{2}/2}^b \frac{dt}{\sqrt{1-t^2}} \\
&= \frac{\pi}{4} + \lim_{b\to 1^-}\int_{\sqrt{2}/2}^b \frac{dt}{\sqrt{1-t^2}}
\end{aligned}$$

With the substitution $u = \sqrt{1-t^2}$ we obtain for this last integral

$$\begin{aligned}
\lim_{b\to 1^-}\int_{\sqrt{2}/2}^b \frac{dt}{\sqrt{1-t^2}} &= \lim_{b\to 1^-}\int_{\sqrt{2}/2}^{\sqrt{1-b^2}} \frac{1}{u}\cdot\frac{-u}{\sqrt{1-u^2}}du \\
&= -\lim_{b\to 1^-}\int_{\sqrt{2}/2}^{\sqrt{1-b^2}} \frac{1}{\sqrt{1-u^2}}du \\
&= -\int_{\sqrt{2}/2}^{0} \frac{1}{\sqrt{1-u^2}}du \\
&= \int_0^{\sqrt{2}/2} \frac{du}{\sqrt{1-u^2}} \\
&= g\left(\tfrac{\sqrt{2}}{2}\right) \\
&= \frac{\pi}{4}
\end{aligned}$$

so that

$$g(1) = g\left(\tfrac{\sqrt{2}}{2}\right) + g\left(\tfrac{\sqrt{2}}{2}\right) = \frac{\pi}{2}$$

□

Corollary $\displaystyle\int_0^{-1}\frac{1}{\sqrt{1-t^2}}dt = -\frac{\pi}{2}$

Proof: Since $\frac{1}{\sqrt{1-t^2}}$ is an even function, for $0 \le a < 1$, we have

$$\int_{-a}^0 \frac{1}{\sqrt{1-t^2}}dt = \int_0^a \frac{1}{\sqrt{1-t^2}}dt$$

and

$$\begin{aligned}
\int_0^{-1}\frac{1}{\sqrt{1-t^2}}dt &= -\int_{-1}^0 \frac{1}{\sqrt{1-t^2}}dt \\
&= -\lim_{a\to 1^-}\int_{-a}^0 \frac{1}{\sqrt{1-t^2}}dt \\
&= -\lim_{a\to 1^-}\int_0^a \frac{1}{\sqrt{1-t^2}}dt \\
&= -\frac{\pi}{2}
\end{aligned}$$

□

We now complete the connection to unit circle trigonometry.

Theorem If $-1 \leq a \leq b \leq 1$ then the arc length of the graph of $y = \sqrt{1-x^2}$ from $x = a$ to $x = b$ is

$$\arcsin b - \arcsin a.$$

Proof: First, note that

$$\frac{d}{dx}\sqrt{1-x^2} = -\frac{x}{\sqrt{1-x^2}}$$

Using the arc length formula and the previous result,

$$\begin{aligned}
\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx &= \int_a^b \sqrt{1+\left(-\frac{x}{\sqrt{1-x^2}}\right)^2}\ dx \\
&= \int_a^b \sqrt{1+\frac{x^2}{1-x^2}}\ dx \\
&= \int_a^b \sqrt{\frac{1}{1-x^2}}\ dx \\
&= \int_a^b \frac{1}{\sqrt{1-x^2}}\ dx \\
&= \arcsin b - \arcsin a
\end{aligned}$$

$\square$

In the particular case that $b = 1$, we have that the arc length s along the upper unit circle from $x = a$ to $x = 1$ is

$$s = \frac{\pi}{2} - \arcsin a$$

Then

$$\begin{aligned}
\cos s &= \cos\left(\frac{\pi}{2} - \arcsin a\right) \\
&= \sin(\arcsin a) \\
&= a
\end{aligned}$$

and

$$\begin{aligned}
\sin s &= \sin\left(\frac{\pi}{2} - \arcsin a\right) \\
&= \cos(\arcsin a) \\
&= \sqrt{1-a^2}
\end{aligned}$$

This shows that a point $P\left(a, \sqrt{1-a^2}\right)$ on the upper unit circle with $-1 \leq a \leq 1$ has coordinates $P(\cos s, \sin s)$ where s is the arc length along the upper unit circle from the point P to $A(1,0)$. This arc length is the definition of the radian measure of angle AOP where $O = (0,0)$.

The connection from geometry-free trigonometry to unit circle trigonometry is complete.