Integration by Parts

Integration by substitution is based on the chain rule. Integration by parts is based on the product rule:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

"Cancelling" the dx term leads to

$$d(uv) = udv + vdu$$

and integrating both sides gives

$$uv = \int udv + \int vdu$$

Re-writing this leads to the

Theorem *(Integration by Parts)*

$$\int u\,dv = u\,v - \int v\,du$$

Remark *Notice that the integration by parts formula does not evaluate the integral but rather transforms it into another integral which is, we hope, easier to evaluate. This should be clarified by the following example:*

Example *Evaluate* $\int x\cos x\,dx.$

Solution: To apply the formula, we need to identify u and dv.
If we take $u = x$ and $dv = \cos x\,dx$, then $du = dx$ and $v = \sin x$.
So:

$$\int x\cos x\,dx = u\,v - \int v\,du = x\sin x - \int \sin x\,dx$$

Now the final integral is easy to perform:

$$\int x\cos x\,dx = x\sin x - (-\cos x) + C = x\sin x + \cos x + C$$

We check by differentiating (using the product rule):
If $f(x) = x\sin x + \cos x + C$ then $f'(x) = (\sin x + x\cos x) - \sin x = x\cos x$ which is the integrand we started with.

Remark *We usually write u, dv, du and v in an array such as:*

$$\begin{array}{ll} u = x & dv = \cos x\,dx \\ du = dx & v = \sin x \end{array}$$

Get in this habit.

Here is another example.

Example *Evaluate* $\int e^x x\,dx.$

Solution: (**First Try**.) The main difficulty is deciding what to take for u and what to take for dv. Suppose we first try

$$u = e^x \qquad dv = x\,dx$$
$$du = e^x\,dx \qquad v = \frac{x^2}{2}$$

Then

$$\int e^x x\,dx = e^x\frac{x^2}{2} - \int e^x\frac{x^2}{2}\,dx.$$

This is correct but notice that the remaining integral is more complicated than the original integral.

(**Second Try**.) So we go back to the start and try

$$u = x \qquad dv = e^x\,dx$$
$$du = dx \qquad v = e^x$$

Then

$$\int e^x x\,dx = xe^x - \int e^x\,dx$$
$$= xe^x - e^x + C.$$

Notice that the resulting integral is easier and has been easily evaluated.

We check by differentiating (using the product rule):

If $f(x) = xe^x - e^x + C$ then $f'(x) = (e^x + xe^x) - e^x = xe^x$ which is the integrand we started with.

Sometimes you need to apply integration by parts more than once:

Example *Evaluate* $\int x^2\cos x\,dx$

Solution: We apply integration by parts with

$$u = x^2 \qquad dv = \cos x\,dx$$
$$du = 2x\,dx \qquad v = \sin x$$

So:

$$\int x^2\cos x\,dx = x^2\sin x - 2\int x\sin x\,dx$$

To evaluate the second integral, we need to apply integration by parts again, this time with

$$u = x \qquad dv = \sin x\,dx$$
$$du = dx \qquad v = -\cos x$$

This gives

$$\int x\sin x\,dx = -x\cos x + \int\cos x\,dx$$
$$= -x\cos x + \sin x + C$$

So

$$\begin{aligned}\int x^2\cos x\,dx &= x^2\sin x - 2\int x\sin x\,dx\\ &= x^2\sin x - 2(-x\cos x + \sin x + C)\\ &= x^2\sin x + 2x\cos x - 2\sin x + C\end{aligned}$$

Remark *Note, the constant $-2C$ has been rewritten as just C. The reason is that "a constant is a constant." So, effectively, we have relabeled $-2C$ as C. This is standard usage.*

There is a shortcut for this two-fold use of the Parts Theorem. This can be seen in the movie "Stand and Deliver."

Layout a table as follows

u	v	sign
x^2	$\cos x$	
$2x$	$\sin x$	$+1$
2	$-\cos x$	-1
0	$-\sin x$	$+1$
	$\cos x$	-1

Now follow the "diagonal product times sign" to get

$$\begin{aligned}&x^2(\sin x)(+1) + 2x(-\cos x)(-1) + 2(-\sin x)(+1) + 0(\cos x)(-1)\\ &= x^2\sin x + 2x\cos x - 2\sin x\end{aligned}$$

and this matches the answer above.

Exercise *Use the shortcut to find* $\int x^3\cos x dx$

Exercise *Use integration by parts to find* $\int \ln x dx$ *(hint:* $u = \ln x \;\; dv = dx$*)*