Chapter 4: Functional Limits and Continuity

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Discussion: Examples of Dirichlet and Thomae

Functional Limits

Combinations of Continuous Functions

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Example 1

Let \( f(x) = \begin{cases} 
1, & \text{if } x \in \mathbb{Q} \\
0, & \text{if } x \notin \mathbb{Q}.
\end{cases} \)

This function is nowhere continuous on \( \mathbb{R} \). (Trust me.)
Modified Dirichlet’s Function

Example 2

Let \( f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases} \)

This function is continuous only at \( c = 0 \). (Trust me.)
Example 3

Let \( f(x) = \begin{cases} 1, & \text{if } x = 0 \\ \frac{1}{n}, & \text{if } x = \frac{m}{n} \in \mathbb{Q}\setminus\{0\} \text{ is in lowest terms with } n > 0 \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases} \)

This function is continuous on the irrationals. It is discontinuous at the rationals. (Trust me.)
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Remark 4

Recall from Chapter 3:

- **Definition:** A point $x$ is a **limit point** of a set $A$ if every $\epsilon$–neighborhood $V_\epsilon(x)$ of $x$ intersects the set $A$ in some point other than $x$.

- **Theorem:** A point $x$ is a limit point of a set $A$ if and only if $x = \lim_{n \to \infty} a_n$ for some sequence $(a_n)$ contained in $A$ satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.

- A point $x \in A$ is **isolated** if it is not a limit point of $A$.

Definition 5

Let $f : A \to \mathbb{R}$, and let $c$ be a limit point of the domain $A$. We say that $\lim_{x \to c} f(x) = L$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$. 

$\epsilon - \delta$ Version of Limit of a Function
Topological Version of Limit of a Function

Remark 6
Recall:

- Let $a > 0$. Then $V_a(x) = \{ y \in \mathbb{R} \mid |x - y| < a \}$. 
- The statement $|x - c| < \delta$ is equivalent to $x \in V_\delta(c)$. 
- The statement $|f(x) - L| < \epsilon$ is equivalent to $f(x) \in V_\epsilon(L)$. 

Definition 7
Let $c$ be a limit point of the domain of $f : A \mapsto \mathbb{R}$. We say $\lim_{x \to c} f(x) = L$ provided that, for every $\epsilon$–neighborhood $V_\epsilon(L)$ of $L$, there exists a $\delta$–neighborhood $V_\delta(c)$ around $c$ with the property that for all $x \in V_\delta(c)$ around $c$ with the property that for all $x \in V_\delta(c)$ different from $c$ (with $x \in A$) it follows that $f(x) \in V_\epsilon(L)$. 


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The Picture

\[
L + \epsilon \\
L \\
L - \epsilon
\]

\[
f(x)
\]
Example

Example 8
Let $f(x) = 5x - 3$. Prove $\lim_{x \to 1} f(x) = 2$.

Example 9
Let $g(x) = x^2 + 1$. Prove $\lim_{x \to -2} g(x) = 5$. 
Sequential Criterion for Functional Limits

**Theorem 10 (Sequential Criterion for Functional Limits)**

Given a function $f : A \mapsto \mathbb{R}$ and a limit point $c$ of $A$, the following are equivalent

- $\lim_{x \to c} f(x) = L$
- For all sequences $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $(x_n) \to c$, it follows that $f(x_n) \to L$.

**Proof:**
Corollary 11

Let \( f \) and \( g \) be functions defined on a domain \( A \subseteq \mathbb{R} \), and assume \( \lim_{x \to c} f(x) = L \) and \( \lim_{x \to c} g(x) = M \) for some limit point \( c \) of \( A \). Then,

1. \( \lim_{x \to c} kf(x) = kL \) for all \( k \in \mathbb{R} \).
2. \( \lim_{x \to c} [f(x) + g(x)] = L + M \).
3. \( \lim_{x \to c} [f(x)g(x)] = LM \).
4. \( \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M} \) provided \( M \neq 0 \).
Corollary 12 (Divergence Criterion for Functional Limits)

Let $f$ be a function defined on $A$, and let $c$ be a limit point of $A$. If there exist two sequences $(x_n)$ and $(y_n)$ in $A$ with $x_n \neq c$ and $y_n \neq c$ for all $n$, and

$$\lim x_n = \lim y_n = c \text{ but } \lim f(x_n) \neq \lim f(y_n),$$

then we can conclude that the functional limit $\lim_{x \to c} f(x)$ does not exist.
Example 13

Let \( f(x) = \sin \left( \frac{1}{x} \right) \). Assume the usual properties of the sine function. Show that \( \lim_{x \to 0} f(x) \) does not exist.
Homework

Pages: 120–122
Problems: 4.2.2, 4.2.5, 4.2.6, 4.2.8, 4.2.9, 4.2.11
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Definition of Continuity

**Definition 14**

- A function $f : A \rightarrow \mathbb{R}$ is **continuous at a point** $c \in A$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - f(c)| < \epsilon$.
- If $f$ is continuous at every point in $A$, then we say that $f$ is **continuous on** $A$.

**Remark 15**

*We can’t shorten the definition of continuity to $\lim_{x \to c} f(x) = f(c)$ because the definition of functional limits requires the point $c$ to be a limit point of $A$. Note that limit points of $A$ need not be elements of $A$. In the definition of continuity, this is not assumed. (Isolated points of $A$ are not limit points).*
Characterizations of Continuity

Theorem 16 (Characterizations of Continuity)

Let $f : A \mapsto \mathbb{R}$, and let $c \in A$ be a limit point of $A$. The function $f$ is continuous at $c$ if and only if any one of the following conditions is met:

- For all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta$ (and $x \in A$) implies $|f(x) - f(c)| < \epsilon$.
- $\lim_{x \to c} f(x) = f(c)$.
- For all $V_{\epsilon}(f(c))$, there exists a $V_{\delta}(c)$ with the property that $x \in V_{\delta}(c)$ (and $x \in A$) implies $f(x) \in V_{\epsilon}(f(c))$.
- If $(x_n) \xrightarrow{n \to \infty} c$ (with $x_n \in A$), then $f(x_n) \xrightarrow{n \to \infty} f(c)$.

Proof:
Corollary 17 (Criterion for Discontinuity)

Let $f : A \mapsto \mathbb{R}$ and let $c \in A$ be a limit point of $A$. If there exists a sequence $(x_n) \subseteq A$ where $(x_n) \xrightarrow{n \to \infty} c$ but such that $f(x_n)$ does not converge to $f(c)$, we may conclude that $f$ is not continuous at $c$. 
Algebraic Continuity Theorem

**Theorem 18 (Algebraic Continuity Theorem)**

Assume $f : A \mapsto \mathbb{R}$ and $g : A \mapsto \mathbb{R}$ are continuous at a point $c \in A$. Then,

- $kf(x)$ is continuous at $c$ for all $k \in \mathbb{R}$.
- $f(x) + g(x)$ is continuous at $c$.
- $f(x)g(x)$ is continuous at $c$.
- $\frac{f(x)}{g(x)}$ is continuous at $c$, provided the quotient is defined.
Example 19

Show the following functions are continuous.

- $f(x) = k$, for any $k \in \mathbb{R}$.
- $g(x) = x$.
- $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, where $a_i \in \mathbb{R}$ for $0 \leq i \leq n$.
- All rational functions (quotients of polynomials) are continuous over their domains.

Remark 20

Fact: $f(x) = \sqrt{x}$ is continuous on its domain. (See book for proof).
Example 21
Show that $f(x) = x \sin \left( \frac{1}{x} \right)$ is continuous at $x = 0$. 
Theorem 22 (Composition of Continuous Functions)

Given \( f : A \mapsto \mathbb{R} \) and \( g : B \mapsto \mathbb{R} \), assume that the range \( f(A) = \{ f(x) \mid x \in A \} \subseteq B \). If \( f \) is continuous at \( c \in A \), and if \( g \) is continuous at \( f(c) \in B \), then

\[
(g \circ f)(x) = g[f(x)]
\]

is continuous at \( c \).

Proof:
Example

Example 23

Note that \( f(x) = \sqrt{x^2 - 7x + 5} \) is continuous on its domain since polynomials and the square root function are both continuous, and the composition of continuous functions remains continuous (on the restricted domain).
Homework

Pages: 126–129
Problems: 4.3.1, 4.3.5, 4.3.6, 4.3.7, 4.3.9, 4.3.11, 4.3.13
Definition

**Remark 24**

Recall: A sequence \((x_n)\) is **bounded** if there exists a number \(M > 0\) such that \(|x_n| \leq M\) for all \(n \in \mathbb{N}\).

**Definition 25**

Let \(f : A \mapsto \mathbb{R}\). We say that \(f\) is **bounded on a set** \(B\), where \(B \subseteq A\) if there exists an \(M > 0\) such that for every sequence \((x_n) \subseteq B\), we have \(|f(x_n)| \leq M\) for all \(n\).
Remark 26

We wish to determine conditions that preserve conditions on sets under continuous functions. That is, if $f$ is continuous and $A$ is open (closed, bounded, compact, etc), does $f(A)$ remain open (closed, etc)?

Example 27

- Consider $f(x) = x^2$ where $f : A = (-1, 1) \mapsto \mathbb{R}$. Is $f(A)$ still open?
- Consider $f(x) = \frac{1}{x^2 + 1} : A = [0, \infty) \mapsto \mathbb{R}$. Is $f(A)$ still closed?
Preservation of Compact Sets

Theorem 28 (Preservation of Compact Sets)

Let $f : A \mapsto \mathbb{R}$ be continuous on $A$. If $K \subseteq A$ is compact, then $f(K)$ is compact.

Proof:
Theorem 29 (The Extreme Value Theorem)

Let $K$ be compact. If $f : K \rightarrow \mathbb{R}$ is continuous, then $f$ attains a maximum and minimum value.

Proof:
Remark 30
Recall: A function \( f : A \mapsto \mathbb{R} \) is **continuous at a point** \( c \in A \) if, for all \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that whenever \( |x - c| < \delta \) (and \( x \in A \)) it follows that \( |f(x) - f(c)| < \epsilon \). A function \( f \) is **continuous on** \( A \) if it is continuous at every point in \( A \).

Definition 31
A function \( f : A \mapsto \mathbb{R} \) is **uniformly continuous on** \( A \) if for every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that \( |x - y| < \delta \) implies \( |f(x) - f(y)| < \epsilon \).

Remark 32
- **For continuity on** \( A \), **we show continuity at each individual point** \( c \), so \( \delta \) may be a function of \( c \).
- **For uniform continuity on** \( A \), the \( \delta \) works simultaneously for all \( c \).
Remark 33
To “break” uniform continuity, it is enough to find a single $\epsilon$ so that no single $\delta$ works for all $c \in A$.

Theorem 34 (Sequential Criterion for Nonuniform Continuity)
A function $f : A \mapsto \mathbb{R}$ fails to be uniformly continuous on $A$ if and only if there exists a particular $\epsilon > 0$ and two sequences $(x_n)$ and $(y_n)$ in $A$ satisfying

$$|x_n - y_n| \xrightarrow{n\to\infty} 0 \text{ but } |f(x_n) - f(y_n)| \geq \epsilon.$$
Example

Example 35
The function $f(x) = \sin \left( \frac{1}{x} \right)$ is continuous on $(0, 1)$, but not uniformly continuous.
Example

Example 36

Let $h(x) = x^2$.

- $h$ is continuous, but not uniformly continuous on $\mathbb{R}$.
- $k(x) = x^2 : [-2, 2] \mapsto \mathbb{R}$ is uniformly continuous on $[-2, 2]$. 
Theorem 37

A function that is continuous on a compact set $K$ is uniformly continuous on $K$.

Proof:
Homework

Pages: 134–135
Problems: 4.4.1, 4.4.4, 4.4.6, 4.4.9, 4.4.10
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Definition 38

- Two nonempty sets $A, B \subseteq \mathbb{R}$ are **separated** if $\overline{A} \cap B$ and $A \cap \overline{B}$ are both empty.

- A set $E \subseteq \mathbb{R}$ is **disconnected** if it can be written as $E = A \cup B$, where $A$ and $B$ are nonempty separated sets.

- A set that is not disconnected is called **connected**.

Example 39

- The sets $A = (1, 3)$ and $B = (3, 5)$ are separated. Hence, $E = A \cup B$ is disconnected.

- The sets $A = (1, 3]$ and $B = (3, 5)$ are not separated. Hence, $E = A \cup B$ is connected.

- The sets $A = (\infty, \sqrt{2}) \cap \mathbb{Q}$ and $B = (\sqrt{2}, \infty) \cap \mathbb{Q}$ are separated. Hence, $\mathbb{Q} = A \cup B$ is disconnected.
Theorem 40
A set $E \subseteq \mathbb{R}$ is connected if and only if, for all nonempty disjoint sets $A$ and $B$ satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with $(x_n)$ in one of $A$ or $B$ and $x$ an element of the other.

Remark 41
The above theorem states that a set is connected if and only if no matter how it is partitioned into two disjoint sets, it at least one of the sets contains a limit point of the other.
Remark 42

The next theorem states only intervals in $\mathbb{R}$ are connected. This includes infinite intervals.

Theorem 43

A set $E \subseteq \mathbb{R}$ is connected if and only if whenever $a < c < b$ with $a, b \in E$, it follows that $c \in E$ as well.
Theorem 44 (Preservation of Connectedness)

Let $f : A \rightarrow \mathbb{R}$ be continuous. If $E \subseteq A$ is connected, then $f(E)$ is connected as well.
Theorem 45 (The Intermediate Value Theorem)

If \( f : [a, b] \mapsto \mathbb{R} \) is continuous and if \( L \) is a real number satisfying \( f(a) < L < f(b) \) or \( f(a) > L > f(b) \), then there exists a point \( c \in (a, b) \) where \( f(c) = L \).

Proof:
Example 46

Show that the function $f(x) = (x - 1)(x - 2)(x - 3)$ contains at least one root in the interval $[0, 4]$. (Use the IVT.)
Homework

Pages: 139 – 140
Problems: 4.5.2, 4.5.3, 4.5.7, 4.5.8
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