5. Series Solutions of Second Order, Linear Equations

3. Series Solutions Near an Ordinary Point, Part II

Consider the differential equation

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0.$$

In the previous section we assumed a series solution of the form $y = \phi(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$. For existence of a solution (from chapter 3) we need $p = Q/P$ and $q = R/P$ to be continuous on an open interval containing x_0. Unfortunately, this is not sufficient to guaranty that we have a power series solution that converges with a positive radius of convergence. To have this, we need a stronger condition.

Definition: $\phi(x)$ is analytic at x_0 if it has a power series centered at x_0 with a positive radius of convergence

This concept of analytic really comes from Complex Analysis, but we will go with the above definition. For example, the common functions like polynomials, sin, cos, and e^x are analytic for all real numbers. The function $\phi(x) = \sqrt[3]{x}$ is continuous at zero, but not analytic.

Theorem 5.3.1 basically says that Q/R and R/P must be analytic in order to guaranty a power series solution.

So What ...

Recall that Taylor Series are given by $\phi(x) = \phi(x_0) + \phi'(x_0)(x - x_0) + \ldots + \frac{\phi^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots$ We can thin of this in another way by saying that if we find a power series solution for a differential equation and call it ϕ as above, then we also know that $\phi^{(n)}(x_0) = n!\, a_n$

Example: If $y'' + e^x y' + y = 0$, with $y(0) = 2$ and $y'(0) = 1$ has a power series solution, $\phi(x)$, then find $\phi'(0)$, $\phi''(0)$, and $\phi^{(5)}(0)$.

Solution: assume a power series solution $y = \phi(x) = \sum_{n=0}^{\infty} a_n x^n$. The center is at zero b/c of the IC's. We then need to determine a_1, a_2, and a_5 in order to answer the question asked. Note that the IC's give us that $a_0 = 2$ and $a_1 = 1$, so we already have that

$$\boxed{\phi'(0) = 1!\, a_1 = 1.}$$

Now let's use Mathematica to find the first few coefficients for the power series solution.

In[1]:= `y[x_] = Sum[a_n * x^n, {n, 0, 7}]`

Out[1]= $a_0 + x\,a_1 + x^2\,a_2 + x^3\,a_3 + x^4\,a_4 + x^5\,a_5 + x^6\,a_6 + x^7\,a_7$

Why stop at $n = 7$? We only need up to a_5, so stopping at $n = 5$ would work, but in this example, how to get more of the coefficients will be demonstrated. We will also need the first few terms of the power series for e^x

In[2]:= `etothex[x_] = Normal[Series[e^x, {x, 0, 7}]]`

Out[2]= $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040}$

Now plug into the differential equation. The right hand side of the equation is

In[3]:= `rhs[x_] = y''[x] + etothex[x] * y'[x] + y[x]`

Out[3]= $a_0 + x\,a_1 + 2\,a_2 + x^2\,a_2 + 6\,x\,a_3 + x^3\,a_3 + 12\,x^2\,a_4 + x^4\,a_4 + 20\,x^3\,a_5 + x^5\,a_5 + 30\,x^4\,a_6 + x^6\,a_6 + 42\,x^5\,a_7 + x^7\,a_7 + \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040}\right)\left(a_1 + 2\,x\,a_2 + 3\,x^2\,a_3 + 4\,x^3\,a_4 + 5\,x^4\,a_5 + 6\,x^5\,a_6 + 7\,x^6\,a_7\right)$

Note that you would need to expand (via extended FOIL) the multiplication of $e^x y'[x]$. Now we need to isolate the various coefficients of powers of x and set them to zero (since the left hand side is the zero function). For instance, the constant coefficient on the right is given by

In[4]:= `Coefficient[rhs[x], x, 0]`

Out[4]= $a_0 + a_1 + 2\,a_2$

The coefficient of *x* on the right is given by

In[5]:= `Coefficient[rhs[x], x, 1]`

Out[5]= $2\,a_1 + 2\,a_2 + 6\,a_3$

To get all of the coefficients on the right we could use

In[6]:= `CoefficientList[rhs[x], x]`

Out[6]=
$$\Big\{a_0 + a_1 + 2\,a_2,\ 2\,a_1 + 2\,a_2 + 6\,a_3,\ \frac{a_1}{2} + 3\,a_2 + 3\,a_3 + 12\,a_4,\ \frac{a_1}{6} + a_2 + 4\,a_3 + 4\,a_4 + 20\,a_5,\ \frac{a_1}{24} + \frac{a_2}{3} + \frac{3\,a_3}{2} + 5\,a_4 + 5\,a_5 + 30\,a_6,\ \frac{a_1}{120} + \frac{a_2}{12} + \frac{a_3}{2} + 2\,a_4 + 6\,a_5 + 6\,a_6 + 42\,a_7,\ \frac{a_1}{720} + \frac{a_2}{60} + \frac{a_3}{8} + \frac{2\,a_4}{3} + \frac{5\,a_5}{2} + 7\,a_6 + 7\,a_7,\ \frac{a_1}{5040} + \frac{a_2}{360} + \frac{a_3}{40} + \frac{a_4}{6} + \frac{5\,a_5}{6} + 3\,a_6 + 8\,a_7,\ \frac{a_2}{2520} + \frac{a_3}{240} + \frac{a_4}{30} + \frac{5\,a_5}{24} + a_6 + \frac{7\,a_7}{2},\ \frac{a_3}{1680} + \frac{a_4}{180} + \frac{a_5}{24} + \frac{a_6}{4} + \frac{7\,a_7}{6},\ \frac{a_4}{1260} + \frac{a_5}{144} + \frac{a_6}{20} + \frac{7\,a_7}{24},\ \frac{a_5}{1008} + \frac{a_6}{120} + \frac{7\,a_7}{120},\ \frac{a_6}{840} + \frac{7\,a_7}{720},\ \frac{a_7}{720}\Big\}$$

So now let's answer the question by finding the values for the a's. First, we know from the IC's that

In[7]:= $a_0 = 2$
$a_1 = 1$

Out[7]= 2

Out[8]= 1

So now we get:

In[9]:= `Solve[`$a_0 + a_1 + 2\,a_2 == 0,\ a_2$`]`

Out[9]= $\left\{\left\{a_2 \to -\frac{3}{2}\right\}\right\}$

In[10]:= $a_2 = -3/2$

Out[10]= $-\frac{3}{2}$

In[11]:= `Solve[`$2\,a_1 + 2\,a_2 + 6\,a_3 == 0,\ a_3$`]`

Out[11]= $\left\{\left\{a_3 \to \frac{1}{6}\right\}\right\}$

In[12]:= $a_3 = 1/6$

Out[12]= $\frac{1}{6}$

In[13]:= `Solve`$\left[\frac{a_1}{2} + 3\,a_2 + 3\,a_3 + 12\,a_4 == 0,\ a_4\right]$

Out[13]= $\left\{\left\{a_4 \to \frac{7}{24}\right\}\right\}$

In[14]:= $a_4 = 7/24$

Out[14]= $\frac{7}{24}$

In[15]:=
```
Solve[a_1/6 + a_2 + 4 a_3 + 4 a_4 + 20 a_5 == 0, a_5]
```

Out[15]= $\left\{\left\{a_5 \to -\frac{1}{40}\right\}\right\}$

In[16]:=
```
a_5 = -1/40
```

Out[16]= $-\frac{1}{40}$

So, we get that $\phi''(0) = 2!\, a_2$ and $\phi^{(5)}(0) = 5!\, a_5$:

In[17]:=
```
2! * a_2
5! * a_5
```

Out[17]= -3

Out[18]= -3

$$\phi'(0) = 1,\ \phi''(0) = -3 \text{ and } \phi^{(5)}(0) = -3.$$

Example 2: $y'' + 3\,x\,y' - \text{Log}[x]\,y = 0$, with $y(1) = 3$ and $y'(1) = -4$. Find the first 5 coefficients of the power series solution centered at 1 (b/c of the IC's).

Solution: First we need to mathematically manipulate the DE so that everything is "centered" at 0. Let $t = x - 1$ or similarly let $u(x+1) = y(x)$, then the IVP become $u'' + 3\,(t+1)\,u' - \text{Log}[t+1]\,u = 0$ with $u(0) = 3$ and $u'(0) = -4$. Then the power series solution centered at 1 for y and the power series solution centered at 0 for u have the same coefficients, so...

To clear the values of the a_n's appears to be difficult. We could quit the kernel, but instead use a different letter.

In[19]:=
```
u[t_] = Sum[b_n t^n, {n, 0, 6}]
R[t_] = Normal[Series[Log[t + 1], {t, 0, 6}]]
```

Out[19]= $b_0 + t\,b_1 + t^2\,b_2 + t^3\,b_3 + t^4\,b_4 + t^5\,b_5 + t^6\,b_6$

Out[20]= $t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - \frac{t^6}{6}$

In[21]:=
```
rhs[x_] = u''[t] + 3 (t + 1) u'[t] - R[t] * u[t]
```

Out[21]= $2\,b_2 + 6\,t\,b_3 + 12\,t^2\,b_4 + 20\,t^3\,b_5 + 30\,t^4\,b_6 + 3\,(1+t)\,\left(b_1 + 2\,t\,b_2 + 3\,t^2\,b_3 + 4\,t^3\,b_4 + 5\,t^4\,b_5 + 6\,t^5\,b_6\right) - \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - \frac{t^6}{6}\right)\left(b_0 + t\,b_1 + t^2\,b_2 + t^3\,b_3 + t^4\,b_4 + t^5\,b_5 + t^6\,b_6\right)$

In[22]:=
```
Coefficient[rhs[t], t, 0]
```

Out[22]= $3\,b_1 + 2\,b_2$

In[23]:=
```
Coefficient[rhs[t], t, 1]
```

Out[23]= $-b_0 + 3\,b_1 + 6\,b_2 + 6\,b_3$

In[24]:=
```
Coefficient[rhs[t], t, 2]
```

Out[24]= $\frac{b_0}{2} - b_1 + 6\,b_2 + 9\,b_3 + 12\,b_4$

Looking at the IC's, we have

In[25]:=
```
b0 = 3
b1 = -4
Solve[3 b1 + 2 b2 == 0, b2]
```

Out[25]= 3

Out[26]= −4

Out[27]= $\{\{b_2 \to 6\}\}$

In[28]:=
```
b2 = 6;
Solve[-b0 + 3 b1 + 6 b2 + 6 b3 == 0,  b3]
```

Out[29]= $\left\{\left\{b_3 \to -\frac{7}{2}\right\}\right\}$

In[30]:=
```
b3 = -7 / 2;
Solve[b0/2 - b1 + 6 b2 + 9 b3 + 12 b4 == 0, b4]
```

Out[31]= $\left\{\left\{b_4 \to -\frac{5}{6}\right\}\right\}$

So the first 5 coefficients are:

$$\boxed{b_0 = 3,\ b_1 = -4,\ b_2 = 6,\ b_3 = -7/2, \text{ and } b_4 = -5/6.}$$