Chapter 2: Solutions of Equations in One Variable

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Overview

Population Example

- Malthusian model of population is $P'(t) = \lambda P(t)$ with $P(0) = P_0$. This has solution: $P(t) = P_0 e^{\lambda t}$.
- If immigration is permited at a constant rate ν, then the model is $p'(t) = \lambda P(t) + \nu \implies P(t) = P_0 e^{\lambda t} + \frac{\nu}{\lambda}\left(e^{\lambda t} - 1\right)$.
- Question: suppose that P is measured in thousands of individuals, t is years, $P_0 = 1,000$, $\nu = 435$ and $P(1) = 1,564$, then find λ.
- That is, find λ where

$$1,564 = 1,000 e^{\lambda} + \frac{435}{\lambda}\left(e^{\lambda} - 1\right)$$

Population Example (Continued)

In this equation, λ appears both inside and outside a transcendental function. Because of this, it is not currently possible to solve for λ. The methods in this chapter are designed to approximate solutions of problems that can not be solved exactly.

Note: If $F(x) = f(x) - a$, then solving $f(x) = a$ for x is equivalent to finding roots (or zeros) of F. That is, $f(p) = a$ if and only if $F(p) = 0$.

Overview

Intermediate Value Theorem

Theorem 1

Let f be a continuous function on the interval $[a, b]$ with $L \in \mathbb{R}$ between $f(a)$ and $f(b)$. Then there exist $c \in [a, b]$ such that $f(c) = L$.

Corollary 2

Let f be a continuous function on the interval $[a, b]$ with $f(a)f(b) < 0$. Then there exists $p \in (a, b)$ such that $f(p) = 0$.

Bisection Algorithm or Binary-search Method

To find an approximation to the solution of $f(x) = 0$ given the continuous function f on the interval $[a, b]$, where $f(a)$ and $f(b)$ have opposite signs:

INPUT endpoints a, b; tolerance TOL; max number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1: Set $i = 1$ and $FA = f(a)$,

Step 2: While $i \leq N_0$ do steps 3-6,

- **Step 3:** Set $p = a + (b - a)/2$ and $FP = f(p)$,
- **Step 4:** If $FP = 0$ or $(b - a)/2 < Tol$ then
  OUTPUT(p) and Stop.
- **Step 5:** Set $i = i + 1$.
- **Step 6:** If $FA \cdot FP > 0$ then $a = p$, $FA = f(p)$,
  else set $b = p$.

Step 7: OUTPUT("Method failed, maximum iterations reached") and Stop.

Example of Bisection Method

Example 3

Find the square root of 10 accurate to 10^{-2}. Hint: find the positive root of $f(x) = x^2 - 10$.

Homework

Homework assignment section 2.1, due: TBA

Overview

Definition and Example

Definition 4

Fixed Point: The number p is a fixed point for a given function g if $g(p) = p$.

Example 5

Determine any fixed points of the function $g(x) = x^2 - 6$

Sufficient Conditions for Existence

Theorem 6

1. *If* $g \in C[a, b]$ *and* $g(x) \in a, b$ *for all* $x \in [a, b]$*, then* g *has at least one fixed point in* $[a, b]$.
2. *If, in addition,* $g'(x)$ *exists on* (a, b) *and a positive constant* $k < 1$ *exists with*

$$|g'(x)| \leq k, \text{ for all } x \in (a, b),$$

then there is exactly one fixed point in $[a, b]$.

Example

Example 7

Show that $g(x) = \cos(x)$ has exactly one fixed point in $[0, \pi/3]$.

Fixed Point Iteration

Suppose we generate a sequence $\{p_n\}$ that converges to p with $p_{n+1} = g(p_n)$ where g is continuous, then p is a fixed point of g.

Proof.

$$p = \lim_{n\to\infty} p_{n+1} = \lim_{n\to\infty} g(p_n) = g\left(\lim_{n\to\infty} p_n\right) = g(p).$$

□

Homework

Homework assignment section 2.2, due: TBA

Overview

Definition and Example

Definition 8

Newton's Method: Suppose $f \in C^1(I)$, where I is an open interval and $\{p_n\}$ be a sequence in I converging to p with $p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$. Then p is a root (or zero) of f.

Example 9

Determine any roots of the function $f(x) = x^2 - 6$

Convergence

Theorem 10

Let $f \in C^2[a,b]$. If $p \in (a,b)$ such that $f(p) = 0$ and $f'(p) \neq 0$, then there exists a $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any initial approximation $p_0 \in [p-\delta, p+\delta]$.

Note: nothing is said about how small δ might be, or the rate of convergence. In practice, for most reasonable problems, Newton's method will either quickly converge or it will be obvious that it will not converge.

Secant Method

Question: What if $f'(x)$ is not readily available?

In this case replace $f'(p_n)$ with an approximation. If p_n is close to p_{n-1}, then

$$f'(p_n) \approx \frac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}}.$$

The Secant method assumes that two initial approximations, p_0 and p_1, are given, then for $n = \mathbb{Z}^+$,

$$p_{n+1} = p_n - \frac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}.$$

Example

Example 11

Use the Secant method with a tolerance of 10^{-4} to approximate $\sqrt{6}$.

Homework

Read: The Method of False Position at the end of the section and then:

Homework assignment section 2.3, due: TBA

Overview

Order of Convergence

Definition 12

Suppose $\{p_n\}$ is a sequence that converges to p, with $p_n \neq p$ for all n. If there exists positive constants λ and α with

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda,$$

then $\{p_n\}$ **converges to p of order α, with asymptotic error constant λ**.

An iterative technique of the form $p_{n+1} = g(p_n)$ is said to be of **order** α if the sequence converges to the solution $p = g(p)$ of order α.

Order of Convergence (cont.)

If $\alpha = 1$ (and $\lambda < 1$), the sequence is **linearly convergent**.

If $\alpha = 2$, the sequence is **quadratically convergent**.

Example 13

Let $p_n = \dfrac{1}{2^n}$. Find the order of convergence to zero.

Theorem

Theorem 14

Let $C \in C[a, b]$ be such that $g(x) \in [a, b]$, for all $x \in [a, b]$. Suppose, in addition, that g' is continuous on (a, b) and that a positive constant $k < 1$ exists with $|g'(x)| \leq k$, for all $x \in (a, b)$. If $g'(p) \neq 0$, then for any number $p_0 \neq p$ in $[a, b]$, the sequence $p_{n+1} = g(p_n)$ converges only linearly to the unique fixed point p in $[a, b]$.

Theorem

Theorem 15

Let p be a solution of the equation $x = g(x)$. Suppose that $g'(p) = 0$ and g'' is continuous with $|g''(x)| < M$ on an open interval I containing p. Then there exists a $\delta > 0$ such that, for $p_0 \in [p - \delta, p + \delta]$, the sequence defined by $p_{n+1} = g(p_n)$ converges at a rate at least quadratically to p. Moreover, for sufficiently large n,

$$|p_{n+1} - p| < \frac{M}{2}|p_n - p|^2.$$

Homework

Read: The part about **Multiple Roots** at the end of the section and then:

Homework assignment section 2.4, due: TBA