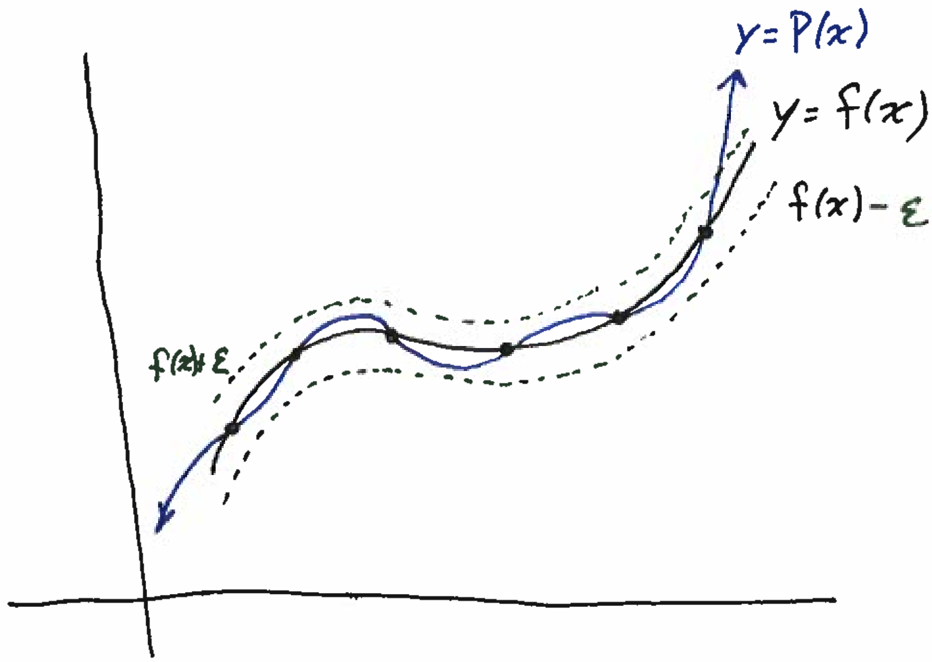
§3.1 Interpolation and Lagrange Polynomials



Let $\{(x_j, y_j)\}_{j=1}^{n}$ be datapoints (from some function or relation) and $P(x)$ be an approximation to the data. Sometimes we let $j = 0, 1, 2, \ldots, n-1$.

Def The error in the approximation is

$$E_{L^p} = \left(\sum_{j=1}^{n} |y_j - P(x_j)|^p \right)^{1/p}$$

Commonly used values of p are 1, 2 and ∞

$E_{L^2} = \left(\sum_{j=1}^{n} |y_j - P(x_j)|^2 \right)^{1/2}$ is called the Euclidean error or square error

$E_{L^\infty} = \max_{j=1,2,\dots,n} \{ |y_j - P(x_j)| \}$ is the max error.

Mostly we will use E_{L^2}.

Background

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

where $a_j \in \mathbb{R}$, $n \in \mathbb{N} = \{0, 1, 2, \dots\}$, is an n^{Th}-degree polynomial in x over the real's.

Weierstrass Approximation theorem

Let $f \in C[a,b]$. $\forall \varepsilon > 0$, $\exists P_n(x) \ni$

$$|f(x) - P_n(x)| < \varepsilon, \ \forall x \in [a,b].$$

So we can approximate continuous functions by polynomials to any accuracy. Note: nothing is said about how big n is.

Polynomials are Unique

$P_n(x) = a_0 + a_1 x + \dots + a_n x^n$, $Q_m(x) = b_0 + b_1 x + \dots + b_m x^m$

then $P_n(x) = Q_m(x)$ iff $n = m$, $a_0 = b_0$, $a_1 = b_1, \dots, a_n = b_n$.

Polynomials can be written in several forms

Standard form: $P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

Taylor form: $f(x) \approx T_n(x) = f(a) + f'(a)(x-a) + \dots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$

Factored form: $P_n(x) = c\,(x-x_1)(x-x_2)\cdots(x-x_n)$
where $x_1, x_2, \dots, x_n$ are the zero's or roots of the polynomial (may be complex).

Interpolation and Lagrange Polynomials

Interpolation: $E = 0$. Interpolating functions pass through all data points (so the data must come from a function).

Lagrange Polynomials:

Example Data: $\{(-1,1), (0,-1), (1,2)\}$ $(n=0,1,2)$
Find the polynomial that interpolates the data.

We have 3 data points so we need to find $P_2(x)$. Why 2^{nd} degree?

Let $L_K = \prod_{j \neq k} \frac{(x - x_j)}{(x_K - x_j)}$

So

$$L_0 = \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{1}{2} x(x-1)$$

$$L_1 = \frac{(x+1)(x-1)}{(0+1)(0-1)} = -(x+1)(x-1)$$

$$L_2 = \frac{(x+1)(x-0)}{(1+1)(1-0)} = \frac{1}{2} x(x+1)$$

Then

$$P_{n-1}(x) = \sum_{j=0}^{n-1} y_j L_j(x)$$

So

$$P_2(x) = (1) L_0(x) + (-1) L_1(x) + (2) L_2(x)$$

$$= \frac{1}{2} x(x-1) + (x+1)(x-1) + (2)(\tfrac{1}{2}) x(x+1)$$

$$= \frac{5}{2} x^2 + \frac{1}{2} x - 1$$

Theorem 3.2 If we have $n+1$ distinct numbers, $\{x_j\}_{j=0}^{n}$ and f is a function whose values are given at these numbers, then $\exists!$ $P(x)$ of degree at most n with $f(x_K) = P(x_K)$ for $K = 0, 1, 2, \dots, n$.

Proof follow the example above.

Theorem 3.3

Suppose $\{x_j\}_{j=0}^{n}$ are distinct numbers in $[a,b]$ and $f \in C^{n+1}[a,b]$. Then for each $x \in [a,b]$, $\exists\ \xi(x) \in (a,b) \ni$

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n),$$

where $P(x)$ is the interpolating polynomial.

In general, we never know the value of $\xi(x)$, so in practice: If $|f^{(n+1)}(x)| \leq M \ \forall x \in (a,b)$, then

$$|f(x) - P(x)| \leq \frac{M}{(n+1)!} \left| \prod_{j=0}^{n} (x - x_j) \right|$$

This gives a bound on the absolute error.

Example suppose the data from the previous example came from a $C^3[-1,1]$ function with $|f^{(3)}(x)| \leq 7$ for $x \in [-1,1]$. then the absolute error is:

$$|f(x) - P_2(x)| \leq \frac{7}{3!} |(x+1)(x-0)(x-1)|, \ \forall x \in [-1,1]$$

Look at how Mathematica does it.