# § 3.3 Divided Differences

Suppose we write a polynomial in the following form:

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots$$
$$+ a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}).$$

If $P_n(x) = f(x)$ at $x_0, x_1, \dots, x_n$, then

$$a_0 = P_n(x_0) = f(x_0).$$

$$f(x_0) + a_1(x_1 - x_0) = P_n(x_1) = f(x_1)$$

$$\Rightarrow \quad a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Notation:

$$f[x_k] = f(x_k)$$

$$f[x_k, x_{k+1}] = \frac{f[x_{k+1}] - f[x_k]}{x_{k+1} - x_k}$$

$$\vdots$$

$$f[x_k, x_{k+1}, \dots, x_{k+m}] =$$

$$\cancel{f[x_{k+1}, x_{k+2}, \dots, x_{k+m}]}$$

$$\frac{f[x_{k+1}, x_{k+2}, \dots, x_{k+m}] - f[x_k, x_{k+1}, \dots x_{k+m-1}]}{x_{k+m} - x_k}$$

Then $\quad a_k = f[x_0, x_1, \dots, x_k]$

**Example** Find an interpolating polynomial passing through $\{(-1,0),(0,1),(1,3),(2,4),(3,7)\}$

| $k$ | $x_k$ | $y_k = f[x_k]$ | $f[x_k, x_{k+1}]$ | $f[x_k, x_{k+1}, x_{k+2}]$ |
|---|---|---|---|---|
| 0 | -1 | 0 | $\frac{1-0}{0-(-1)} = 1$ | $\frac{2-1}{1-(-1)} = \frac{1}{2}$ |
| 1 | 0 | 1 | $\frac{3-1}{1-0} = 2$ | $\frac{1-2}{2-0} = -\frac{1}{2}$ |
| 2 | 1 | 3 | $\frac{4-3}{2-1} = 1$ | $\frac{3-1}{3-1} = 1$ |
| 3 | 2 | 4 | $\frac{7-4}{3-2} = 3$ | |
| 4 | 3 | 7 | | |

| $f[x_k, x_{k+1}, x_{k+2}, x_{k+3}]$ | $f[x_k, \dots, x_{k+4}]$ |
|---|---|
| $\frac{-\frac{1}{2}-\frac{1}{2}}{2-(-1)} = -\frac{1}{3}$ | $\frac{\frac{1}{2}-(-\frac{1}{3})}{3-(-1)} = \frac{5}{24}$ |
| $\frac{1-(-\frac{1}{2})}{3-0} = \frac{1}{2}$ | |

So $a_0 = 0$, $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = -\frac{1}{3}$, $a_4 = \frac{5}{24}$ and the interpolating polynomial is

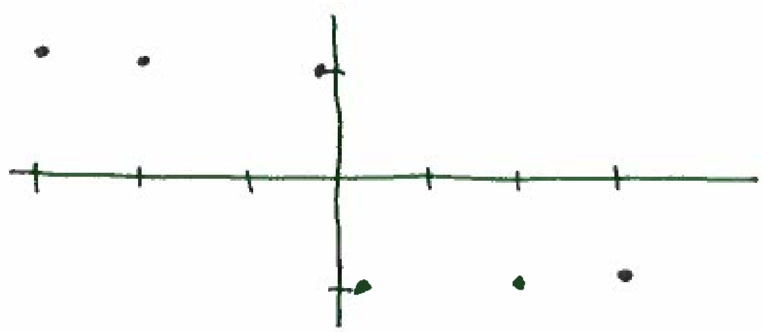
$$P_4(x) = 0 + 1(x+1) + \tfrac{1}{2}(x+1)(x) - \tfrac{1}{3}(x+1)(x)(x-1) + \tfrac{5}{24}(x+1)(x)(x-1)(x-2)$$

Note: this method uses "Backward" differences. Read the section to see "Forward" and "Centered" diff. Also note that code for this is on my website.

**Example** Use NDD.m to approximate the following data with a polynomial, then use a program to graph the polynomial and the data.

data $= \{(-3,1),(-2,1),(-0.1,1),(0.1,-1),(2,-1),(3,-1)\}$

Note: the data looks like



a step function. The polynomial will not!