§3.~~4~~ Hermite Interpolation

Osculating polynomials – generalization of Taylor and Lagrange polynomials.

Given $n+1$ data points $\{x_k\}_{k=0}^{n}$ in $[a,b]$ and non-negative integers $m_0, m_1, \ldots, m_n$ and $m = \max\{m_k\}_{k=0}^{n}$ the osculating polynomial approximating $f \in C^m[a,b]$ at x_k for $k=0,1,2,\ldots,n$, is the polynomial of least degree that agrees with $f^{(j)}(x)$ at x_k for $k=0,1,\ldots,n$ and $j=0,1,\ldots,m_k$

The degree of the Polynomial is at most

$$M = \sum_{k=0}^{n} m_k + n$$

If $P(x)$ is the osculating polynomial approximating f then

$$\frac{d^j P}{dx^j}(x_k) = \frac{d^j f}{dx^j}(x_k)$$

for $k=0,1,2,\ldots,n$ and $j=0,1,\ldots,m_k$

If $n=0$, then $P(x)$ is a Talyor Polynomial

If $m_k=0$, $\forall k$, then $P(x)$ is a Lagrange Polymomial.

Let $$L_{n,k}(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}$$

and $$H_{n,j}(x) = \left(1 - 2(x - x_j) L'_{n,j}(x_j)\right) L^2_{n,j}(x)$$

and $$\hat{H}_{n,j}(x) = (x - x_j) L^2_{n,j}(x)$$

Then $$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j) H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_{n,j}(x)$$

agrees with f and f' on $x_0, x_1, \ldots, x_n$.

Example

$f(x) = \cos(x)$, $x_0 = \frac{\pi}{6}$, $x_1 = \frac{\pi}{4}$, $x_2 = \frac{\pi}{3}$

Note $n = 2$.

$$L_{2,0}(x) = \frac{(x - \frac{\pi}{4})(x - \frac{\pi}{3})}{(\frac{\pi}{6} - \frac{\pi}{4})(\frac{\pi}{6} - \frac{\pi}{3})}$$

$$L_{2,1}(x) = \frac{(x - \frac{\pi}{6})(x - \frac{\pi}{3})}{(\frac{\pi}{4} - \frac{\pi}{6})(\frac{\pi}{4} - \frac{\pi}{3})}$$

$$L_{2,2}(x) = \frac{(x - \frac{\pi}{6})(x - \frac{\pi}{4})}{(\frac{\pi}{3} - \frac{\pi}{6})(\frac{\pi}{3} - \frac{\pi}{4})}$$

$$H_{2,0}(x) = \left(1 - 2(x - \tfrac{\pi}{6}) L'_{2,0}(\tfrac{\pi}{6})\right) L^2_{2,0}(x) \qquad \text{degree } 5$$

$$H_{2,1}(x) = \left(1 - 2(x - \tfrac{\pi}{4}) L'_{2,1}(\tfrac{\pi}{4})\right) L^2_{2,1}(x)$$

$$H_{2,2}(x) = \left(1 - 2(x - \tfrac{\pi}{4}) L'_{2,2}(\tfrac{\pi}{3})\right) L^2_{2,2}(x)$$

$$\hat{H}_{2,0}(x) = \left(x - \frac{\pi}{6}\right) L_{2,0}^{2}(x)$$

$$\hat{H}_{2,1}(x) = \left(x - \frac{\pi}{4}\right) L_{2,1}^{2}(x)$$

$$\hat{H}_{2,2}(x) = \left(x - \frac{\pi}{3}\right) L_{2,2}^{2}(x)$$

degree 5

then

$$H_5(x) = \frac{\sqrt{3}}{2} H_{2,0}(x) + \frac{1}{\sqrt{2}} H_{2,1}(x) + \frac{1}{2} H_{2,2} - \frac{1}{2}\hat{H}_{2,0}(x) - \frac{1}{\sqrt{2}}\hat{H}_{2,1}(x) - \frac{\sqrt{3}}{2}\hat{H}_{2,2}(x)$$

Hermite Polynomials Using Divided Differences

$$H_{2n+1}(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0, \ldots, z_k](x - z_0)(x - z_1)\cdots(x - z_{k-1})$$

where

z	$f(z)$	First divided differences	2^{nd} divided differences
$z_0 = x_0$	$f[z_0] = f(z_0)$		
		$f[z_0, z_1] = f'(x_0)$	$f[z_0, z_1, z_2] = \dfrac{f[z_1, z_2] - f[z_0, z_1]}{z_2 - z_0}$
$z_1 = x_0$	$f[z_1] = f(z_1)$		
		$f[z_1, z_2] = \dfrac{f[z_2] - f[z_1]}{z_2 - z_1}$	$\vdots$
$z_2 = x_1$	$f[z_2] = f(z_2)$		
		$f[z_2, z_3] = f'(x_1)$	
$z_3 = x_1$	$f[z_3] = f(z_3)$		
		$f[z_3, z_4] = \dfrac{f[z_4] - f[z_3]}{z_4 - z_3}$	
$z_4 = x_2$	$f[z_4] = f(z_4)$		
		$f[z_4, z_5] = f'(x_2)$	
$z_5 = x_2$	$f[z_5] = f(z_5)$		